Source: Lehner, M., 2012: Observations and large-eddy simulations of the thermally driven cross-basin circulation in a small, closed basin. Ph.D. Thesis, University of Utah. (APPENDIX B)

Extracting terms of the horizontal momentum and thermodynamic equations in the WRF model code

The ARW model equations and the time integration scheme are described in detail by Skamarock et al. (2008) and Skamarock and Klemp (2008). Prognostic model equations for horizontal momentum and potential temperature as well as the time integration scheme are summarized here based on these descriptions prior to detailing the code modifications that are necessary to extract the individual tendency terms. The following model equations have been simplified by neglecting terms that arise from forcing mechanisms that are not active in the idealized simulation described in Chapter **??**.

1 Model equations

In the ARW the non-hydrostatic, fully compressible Euler equations are formulated in flux form using a terrain-following pressure coordinate in the vertical, which is defined as

$$\eta = \frac{p_h - p_{ht}}{\mu}, \tag{1}$$

where μ is the dry air mass per unit area within a model column, which is defined as $\mu = p_{hs} - p_{ht}$. The pressure variables p_h, p_{ht}, and p_{hs} are the hydrostatic pressure of the dry atmosphere, the hydrostatic pressure of the dry atmosphere at the top of the model domain, and the hydrostatic pressure of the dry atmosphere at the surface, respectively. In this coordinate system the horizontal momentum equations and the thermodynamic equation are written as

$$\frac{\partial U}{\partial t} + (\nabla \cdot \mathbf{V}u) + \mu\alpha\frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial x} = F^U_{sgs} + F^U_{diff6} + F^U_{rayl} \tag{2}$$

$$\frac{\partial V}{\partial t} + (\nabla \cdot \mathbf{V}v) + \mu\alpha\frac{\partial p}{\partial y} + \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \phi}{\partial y} = F^V_{sgs} + F^V_{diff6} + F^V_{rayl} \tag{3}$$

$$\frac{\partial \Theta}{\partial t} + (\nabla \cdot \mathbf{V}\theta) = F^\Theta_{rad} + F^\Theta_{sgs} + F^\Theta_{diff6} + F^\Theta_{rayl}, \tag{4}$$

where t is time, x and y are the horizontal coordinates, p is pressure, ϕ is geopotential, α is the specific volume of air including moisture, α_d is the specific volume of dry air, u and v are the horizontal wind components in x and y direction, respectively, and $\mathbf{V}$ is the three-element wind vector (U, V, Ω) with Ω being the vertical velocity in η coordinates. Variables U, V, Ω, and Θ are coupled to the dry air mass and momentum variables U, V, and Ω are additionally coupled to map-scale factors. Since the map-scale factors are 1 in our idealized simulation, which does not include projections onto the sphere, we will not write the map-scale factors, thus greatly simplifying the equations. However, it must kept in mind that for real-case simulations some of the tendency terms in the model code are coupled to map-scale factors. Variables U, V, Ω and Θ are thus defined as $U = \mu u$, $V = \mu v$, $\Omega = \mu\frac{\partial \eta}{\partial t}$, and $\Theta = \mu\theta$. The second term on the left-hand side of (2)–(4) is advection and the third and fourth terms

in the momentum equations are the horizontal pressure-gradient force in η coordinates. Terms F^U, F^V, and F^Θ are the forcing terms from various parameterizations for U, V, and Θ, respectively. They result from the radiation parameterization (subscript rad); from the subgrid-scale turbulence scheme (sgs), whose horizontal and vertical components are calculated individually in the model; from the 6^{th}-order diffusion scheme ($diff6$); and from Rayleigh damping ($rayl$). Equations (2)–(4) include only those terms that arise from forcing mechanisms that are active in the semi-idealized model simulation described in Chapter **??**; all other terms are neglected for simplification. These other terms include, for example, Coriolis forcing terms or curvature forcing terms due to spherical projections in the horizontal momentum equations and cumulus parameterizations or microphysics forcing terms in the thermodynamic equation.

To reduce truncation errors and machine-rounding errors thermodynamic model variables are defined as perturbations from a hydrostatically balanced base state, that is, $p' = p - \bar{p}(\bar{z})$, $\phi' = \phi - \bar{\phi}(\bar{z})$, $\mu' = \mu - \bar{\mu}(x, y)$, and $\alpha' = \alpha - \bar{\alpha}(\bar{z})$ with $\bar{z}(x, y, \eta)$. Using the perturbation variables the horizontal momentum equations (2) and (3) become

$$\frac{\partial U}{\partial t} + (\nabla \cdot \mathbf{V}u) + \mu\alpha' \frac{\partial \bar{p}}{\partial x} + \mu\alpha \frac{\partial p'}{\partial x} + \frac{\alpha}{\alpha_d}\left(\mu \frac{\partial \phi'}{\partial x} + \frac{\partial p'}{\partial \eta}\frac{\partial \phi}{\partial x} - \mu' \frac{\partial \phi}{\partial x}\right) = \tag{5}$$
$$F^U_{sgs} + F^U_{diff6} + F^U_{rayl}$$

$$\frac{\partial V}{\partial t} + (\nabla \cdot \mathbf{V}v) + \mu\alpha' \frac{\partial \bar{p}}{\partial y} + \mu\alpha \frac{\partial p'}{\partial y} + \frac{\alpha}{\alpha_d}\left(\mu \frac{\partial \phi'}{\partial y} + \frac{\partial p'}{\partial \eta}\frac{\partial \phi}{\partial y} - \mu' \frac{\partial \phi}{\partial y}\right) = \tag{6}$$
$$F^V_{sgs} + F^V_{diff6} + F^V_{rayl}$$

2 Time integration

The ARW model uses a third-order Runge-Kutta time integration scheme, which advances the prognostic variables in three sub-steps from the current time step (e.g., U_t) to the next time step (e.g., $U_{t+\Delta t}$), where Δt is the time step:

$$\text{Step 1:} \qquad U_{RK1} = U_t + \frac{\Delta t}{3} F(U_t) \tag{7}$$

$$\text{Step 2:} \qquad U_{RK2} = U_t + \frac{\Delta t}{2} F(U_{RK1}) \tag{8}$$

$$\text{Step 3:} \qquad U_{t+\Delta t} = U_t + \Delta t F(U_{RK2}). \tag{9}$$

Terms $F(U)$ are total forcing terms, that is, the sum of all terms in (5) except for the time derivative.

Fast propagating acoustic modes are integrated using a time-split scheme, with a time step $\Delta\tau$ that is smaller than the Runge-Kutta time step Δt. The number of acoustic time steps within a Runge-Kutta sub-step and the length of $\Delta\tau$ varies among the three Runge-Kutta sub-steps; the number for the last sub-step is defined by the user. In the time-split scheme for the acoustic modes new perturbation variables are defined as the deviation of the variable at the current time (or the perturbation variable for the thermodynamic variables as described above) from the variable at the latest Runge-Kutta sub-step, for example, $U'' = U_t - U_t$ during sub-step 1, $\Theta'' = \Theta_t - \Theta_{RK1}$ during sub-step 2, or $\mu'' = \mu'_t - \mu'_{RK2}$ during sub-step 3. The individual tendency terms in (4)–(6) are calculated during every Runge-Kutta sub-step and remain constant throughout all small time steps within one Runge-Kutta sub-step. The small-time-step perturbation variables are then advanced during every acoustic time step from, for

example, U''_τ to $U''_{\tau+\Delta\tau}$ and the additional terms resulting from substituting the new small-time-step perturbation variables in (4)–(6) are added as a correction to the forcing from the Runge-Kutta sub-step. For the horizontal momentum equations and the thermodynamic equation this is a correction of the pressure-gradient force and the advection, respectively.

3 Extracting tendency terms

To output the individual terms of the horizontal momentum and thermodynamic equations a new variable needs to be created for every forcing term, which will contain the 3D tendency array. The new variables are created in the Registry (`/Registry/Registry.EM`) as `state` variables so that they can be included in the standard model output. Note that u tendency terms are located at u points on the staggered grid, v terms at v points, and θ terms at mass points so that the appropriate staggering option needs to be set in the Registry.[1] Seven new variables are needed for the θ tendency terms in this simplified, semi-idealized simulation (radiation, horizontal and vertical diffusion, advection, 6^{th}-order diffusion, Rayleigh damping, and acoustic-time-step correction term) and 7 variables for the u and v tendency terms (horizontal and vertical diffusion, advection, pressure-gradient force, 6^{th}-order diffusion, Rayleigh damping, and acoustic-time-step correction term). Additional arrays are necessary if the simulation is less idealized or uses other parameterizations, for example, arrays for the Coriolis terms for u and v, the microphysics terms or cumulus parameterization terms for θ, or the curvature terms for u and v in real-case applications, which arise due to map projections.

Tendencies from physics parameterizations (radiation and horizontal and vertical diffusion) are calculated during the first Runge-Kutta sub-step and are added to the model variables `ru_tendf`, `rv_tendf`, and `t_tendf`. All other tendencies (advection, horizontal pressure-gradient force, 6^{th}-order diffusion, and Rayleigh damping) are calculated during every Runge-Kutta sub-step and are added to the variables `ru_tend`, `rv_tend`, and `t_tend`. Physics tendencies `ru_tendf`, `rv_tendf`, and `t_tendf` are then added to non-physics tendencies `ru_tend`, `rv_tend`, and `t_tend`, respectively, so that the latter contain the full tendencies before they are used to advance the prognostic variables. Individual tendency terms can thus be extracted by tracking the above variables in subroutines `first_rk_step_part2` and `rk_tendency`. This is shown below for advection. Advection is calculated in subroutines `advect_u`, `advect_v`, `advect_w`, and `advect_scalar`, which are called from subroutine `rk_tendency`. We define three additional auxiliary variables `utend_aux`, `vtend_aux`, and `ttend_aux` in subroutine `rk_tendency` with dimensions (`ims:ime,kms:kme,jms:jme`). Before the call to the advection subroutines the current tendency arrays are saved in the new auxiliary arrays:

```
DO i = ims, ime
DO k = kms, kme
DO j = jms, jme
  utend_aux(i,k,j) = ru_tend(i,k,j)
  vtend_aux(i,k,j) = rv_tend(i,k,j)
  ttend_aux(i,k,j) = t_tend(i,k,j)
ENDDO
ENDDO
ENDDO
```

[1] For a description of the Registry see the ARW User Guide. The current version can be downloaded from `http://www.mmm.ucar.edu/wrf/users/pub-doc.html`

After the call to the advection subroutines the advection tendencies can then be calculated as the difference between the updated tendency arrays and the old tendency arrays stored in the auxiliary variables:

```
DO i = ims, ime
DO k = kms, kme
DO j = jms, jme
  utend_adv(i,k,j) = ru_tend(i,k,j) - utend_aux(i,k,j)
  vtend_adv(i,k,j) = rv_tend(i,k,j) - vtend_aux(i,k,j)
  ttend_adv(i,k,j) = t_tend(i,k,j) - ttend_aux(i,k,j)
ENDDO
ENDDO
ENDDO
```

Velocity and potential temperature tendencies are coupled to the total dry air mass μ (model variable `mut` at mass points, `muu` at u points, and `muv` at v points). The individual tendency terms thus need to be decoupled by dividing them by μ to get tendencies in units m s^{-2} and K s^{-1}, respectively. All terms of the thermodynamic equation are coupled to `mut` except for microphysics tendencies, which are uncoupled. Terms of the horizontal momentum equations are generally coupled to `muu` and `muv`, respectively. The dry air mass at mass points (`mut`) is interpolated linearly to compute `muu` and `muv`. Tendency terms of the horizontal momentum equations resulting from physics parameterizations (PBL, horizontal and vertical diffusion parameterizations), however, are coupled to `mut` because physics parameterizations are performed on an unstaggered Arakawa A grid. Here, we used only μ at mass points to decouple all the variables. The difference between `mut` and `muu` amounts generally to less than 1% even over the sloping basin sidewalls so that errors induced by this simplification are negligible.

Prognostic variables are advanced during every small or acoustic time step in subroutines `advance_uv`, `advance_mu_t`, and `advance_w`, which are called from subroutine `solve_em`. The correction terms arising from the use of perturbation variables in the time-split scheme are also calculated in these subroutines. Since the variables are advanced finally from time t to time $t + \Delta t$ during the last Runge-Kutta sub-step, we are only interested in the correction terms of Runge-Kutta step 3. For example, U'' (variable `grid%u_2` in subroutine `solve_em`) is advanced in subroutine `advance_uv`:

```
grid%u_2(i,k,j) = grid%u_2(i,k,j) +                          &
                  dts_rk * grid%ru_tend(i,k,j) + sound_corr
```

The variable names in the above code line were changed from their names in subroutine `advance_uv` to match the names in subroutine `solve_em` and the variable `sound_corr` replaces the more complex expression of the acoustic time step correction term. This calculation is performed at every small time step. The total acoustic-time-step correction (`utend_sound`) is thus the sum of all `sound_corr` terms during Runge-Kutta step 3, which can be calculated in subroutine `solve_em` after the call to subroutine `advance_uv`:

```
IF ( rk_step .eq. 3 ) THEN
  DO i = ims, ime
  DO k = kms, kme
  DO j = jms, jme
    grid%utend_sound(i,k,j) = grid%utend_sound(i,k,j) +       &
              ( grid%u_2(i,k,j) - utend_aux(i,k,j) -          &
```

```
                dts_rk * grid%ru_tend(i,k,j) )
  ENDDO
  ENDDO
  ENDDO
ENDIF
```

Variable `utend_aux` is again an auxiliary array that contains `grid%u_2` from just before the call to subroutine `advance_uv` and `dts_rk` is the acoustic time step. `grid%utend_sound` must be initialized with 0 before the loop over all small time steps. Acoustic time step correction terms for v and θ can be determined identically; V'' is also advanced in subroutine `advance_uv` and Θ'' is advanced in subroutine `advance_mu_t`.

In addition to u, v, and θ tendencies, total geopotential ϕ and dry air mass μ tendencies are needed for the calculation of pressure tendencies (section **??**). The geopotential tendency from the large Runge-Kutta step is stored in variable `ph_tend` in subroutine `solve_em`, which, however, is defined as an `i1` variable in the Registry and, thus, cannot be output directly. The geopotential is advanced in subroutine `advance_w` and the small time step correction term can be determined similarly to u, with the exception that `ph_tend` is coupled to `mut`, whereas ϕ itself (variable `grid%ph_2`) is not. Therefore, the tendency from the Runge-Kutta time step needs to be decoupled before subtracting it from the advanced variable:

```
grid%ph_tend_sound(i,k,j) = grid%ph_tend_sound(i,k,j) +       &
             ( grid%ph_2(i,k,j) - ph_tend_aux(i,k,j) -        &
               dts_rk * ph_tend(i,k,j) / grid%mut(i,j) )
```

Air mass is advanced in subroutine `advance_mu_t` and the total tendency, that is, the sum of the Runge-Kutta tendency plus the small time step correction, is stored in variable `grid%mudf`. The total tendency (`mu_tend`) can thus be calculated easily by adding the two components after the call to subroutine `advance_mu_t`:

```
grid%mu_tend(i,j) = grid%mu_tend(i,j) + grid%mudf(i,j) * dts_rk
```

Note that air mass variables are only two-dimensional.

References

Skamarock, W. C., and J. B. Klemp, 2008: A time-split nonhydrostatic atmospheric model for weather research and forecasting applications. *J. Comput. Phys.*, **227**, 3465–3485.

Skamarock, W. C., and Coauthors, 2008: A description of the Advanced Research WRF version 3. Tech. Rep. NCAR/TN-475+STR. 113 pp.