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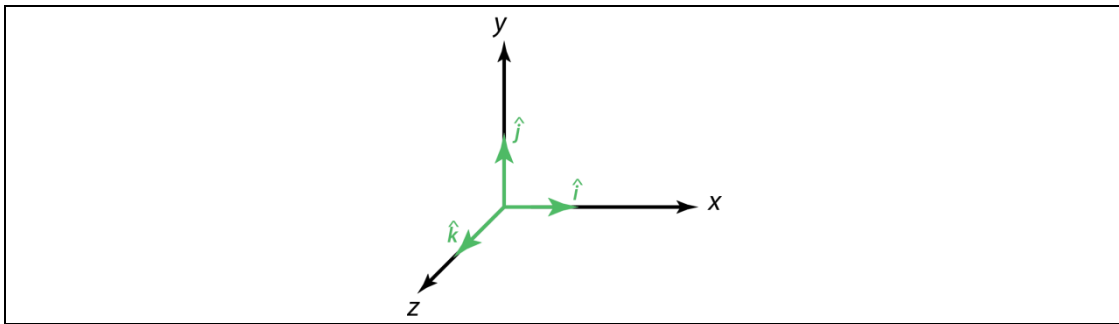
**Lectures 1 & 2 - Review of Vector Calculus**


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This is a quick review of some of the major concepts in vector calculus that is used in this class.

In this appendix I use the following notation.

- Vectors are denoted with an arrow over the top of the variable. E.g.,  $\vec{v}$ .
- Unit vectors, vectors with magnitude = 1, are denoted with a carat over the top of the variable. E.g.,  $\hat{k}$ .
- I use the unit vectors,  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$ , as the unit vectors pointing in the directions of the x-, y- and z-coordinate axes respectively.



- I generally write vectors out as:

$$\vec{u}(x, y, z) = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$$

Here,  $u_1$  is the  $x$ -component of the vector  $\vec{u}$ . Similarly,  $u_2$  is the  $y$ -component and  $u_3$  is the  $z$ -component. It is also common to denote the vector components in Cartesian coordinates as follows:

$$\vec{u}(x, y, z) = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$$

Where we use the subscripts  $x$ ,  $y$ , and  $z$  instead of  $1$ ,  $2$  and  $3$ .

As another alternate, it is sometimes more convenient to write this out in matrix notation as:

$$\vec{u}(x, y, z) = [u_1, u_2, u_3]$$

- The magnitude of a vector is denoted as:  $|\vec{u}|$ , where:

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

- Although we won't define them until next week, tensors are denoted with two solid lines over the top of the variable. E.g.,  $\bar{\bar{\sigma}}$ .

## 1. Fields

**Fields** – A field is a continuous function that returns a number, or set of numbers, for every point in space and time ( $x, t$ ). There are three basic types of fields that we will deal with.

- 1) **Scalar Fields** – A scalar field  $f(x, y, z, t)$  returns a single number for every point in space and time.

*Examples are:* temperature, density, elevation.

These are the easiest fields to visualize. For example, the following MATLAB script shows three ways in which we can represent a scalar field. Let's just do this in 2D for simplicity and let the function resemble some peaks:

$$\text{elevation}(x, y) = (x^2 + 3y^2)e^{1-(x^2+y^2)}$$

```
clear all

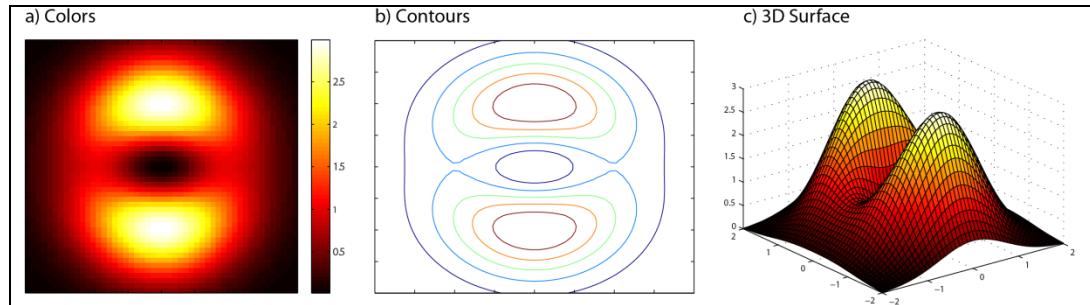
%set up the grid
xgrid=linspace(-pi/4,pi/4,50);
ygrid=linspace(-pi/2,pi/2,50);
[x,y]=meshgrid(xgrid,ygrid);

%calculate the elevations
elev = (x.^2 + 3.*y.^2) .*exp(1.0-(x.^2 + y.^2));

%figure 1 - just plot as colors
figure(1)
colormap(hot)
set(gcf, 'PaperOrientation', 'landscape', 'PaperPosition', ...
    [.25 .25 10.5 8], 'PaperType', 'usletter');
imagesc(xgrid,ygrid,elev)
set(gca, 'YDir', 'normal')
xlabel('X')
ylabel('Y')
title('Elevation')
colorbar

%figure 2 - plot contour lines
figure(2)
set(gca, 'YDir', 'normal')
[C,h]=contour(xgrid,ygrid,elev,2);

%figure 3 - plot 3D surface
figure(3)
colormap(hot)
set(gca, 'YDir', 'normal')
surf(xgrid,ygrid,elev)
```



- 2) **Vector Fields** – A vector field  $\vec{F}(x,y,z,t)$  returns a vector for every point in space and time and is readily visualized as a field of arrows.

**Examples are:** velocity, elastic displacement, electric or magnetic fields.

As an example, let's consider the vector field described as follows:

$$\vec{F}(x, y) = (x + y)\hat{i} + y\hat{j}$$

This equation describes a vector for each position. For example, at  $x = 0, y = 1$ :

$$\vec{F}(0,1) = (0 + 1)\hat{i} + 1\hat{j}$$

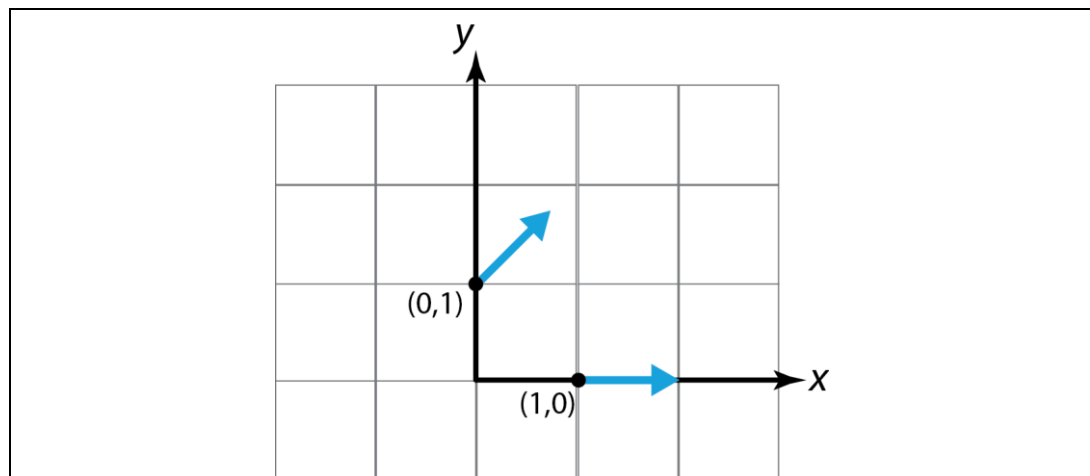
$$\Rightarrow \vec{F}(0,1) = \hat{i} + \hat{j}$$

Or, at  $x = 1, y = 0$ :

$$\vec{F}(1,0) = (1 + 0)\hat{i} + 0\hat{j}$$

$$\Rightarrow \vec{F}(1,0) = \hat{i}$$

We could now draw the vectors for these two positions by hand:



Now, in order to see what the vector field looks like, we need to continue in filling in vectors. This can be time consuming, so let's do it again using MATLAB.

```

% MATLAB script for plotting vector fields
clear all

%set up the grid space
%-----%
xmin = -10;      %minimum x-axis
xmax = 10;       %maximum x-axis
nx   = 11;       %number of points in x-direction

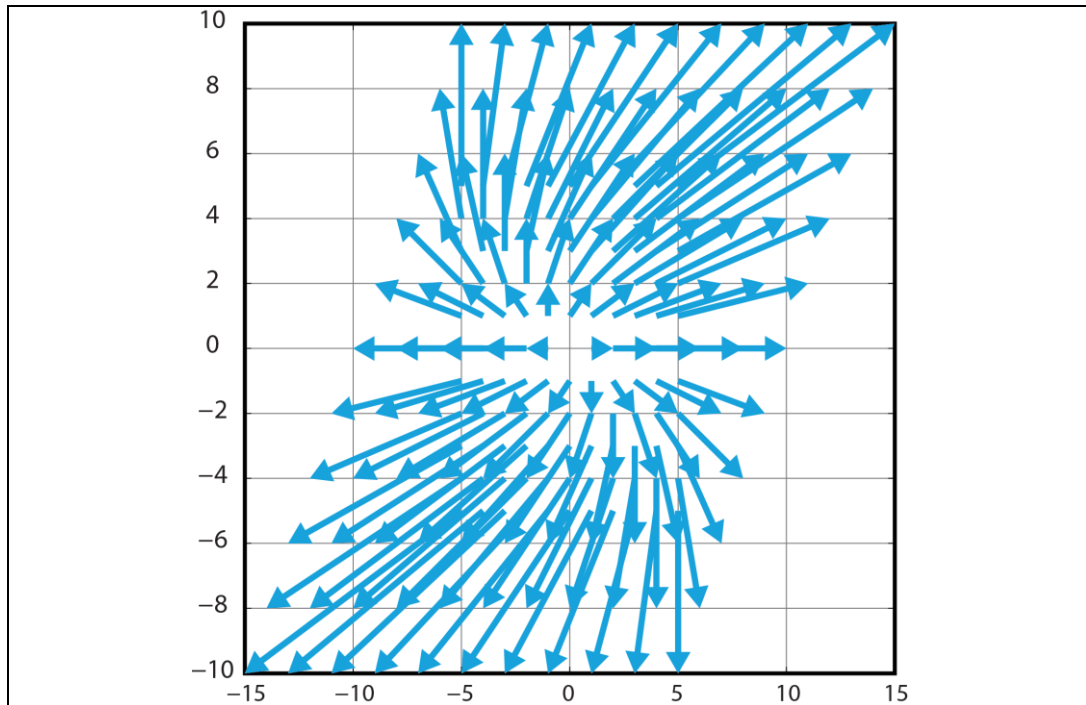
ymin = -5;       %minimum y-axis
ymax = 5;        %maximum y-axis
ny   = 25;       %number of points in y-direction
%-----%

%set up the grid in a mesh
%-----%
xgrid = linspace(xmin,xmax,nx);
ygrid = linspace(ymin,ymax,ny);
[Xmesh,Ymesh]=meshgrid(xgrid,ygrid);
%-----%

%calculate the vector field
% example for  $F(x,y) = (x+y)i + (y)j$ 
%-----%
for i=1:length(xgrid)
    for j=1:length(ygrid)
        x = Xmesh(j,i);      %position, x
        y = Ymesh(j,i);      %position, y
        Vx(j,i) = x + y;     %X-component of vector
        Vy(j,i) = y;         %Y-component of vector
    end
end
%-----%

% plot the vector field with the quiver statement
%-----%
quiver(Xmesh,Ymesh,Vx,Vy,1,'filled','b')
axis('square')
xlabel('x')
ylabel('y')
grid on
%-----%

```



- 3) **Tensor Fields** – Not everything can be represented as a scalar or vector, but can be represented as tensors instead. E.g., a second rank tensor field  $\bar{D}(x,y,z,t)$  can be visualized as a field of ellipsoids (3 orthogonal vectors for each point in space and time).

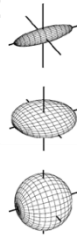
**Examples are:** stress, strain, strain rate, elasticity.

An example (from: [https://graphics.ethz.ch/teaching/former/scivis\\_07/Notes/Slides/09-tensorField.pdf](https://graphics.ethz.ch/teaching/former/scivis_07/Notes/Slides/09-tensorField.pdf))

**Symmetric** 3D tensors have real eigenvalues and orthogonal eigenvectors → they can be represented by **ellipsoids**.

Three types of anisotropy:

- **linear anisotropy**
- **planar anisotropy**
- **isotropy** (spherical)



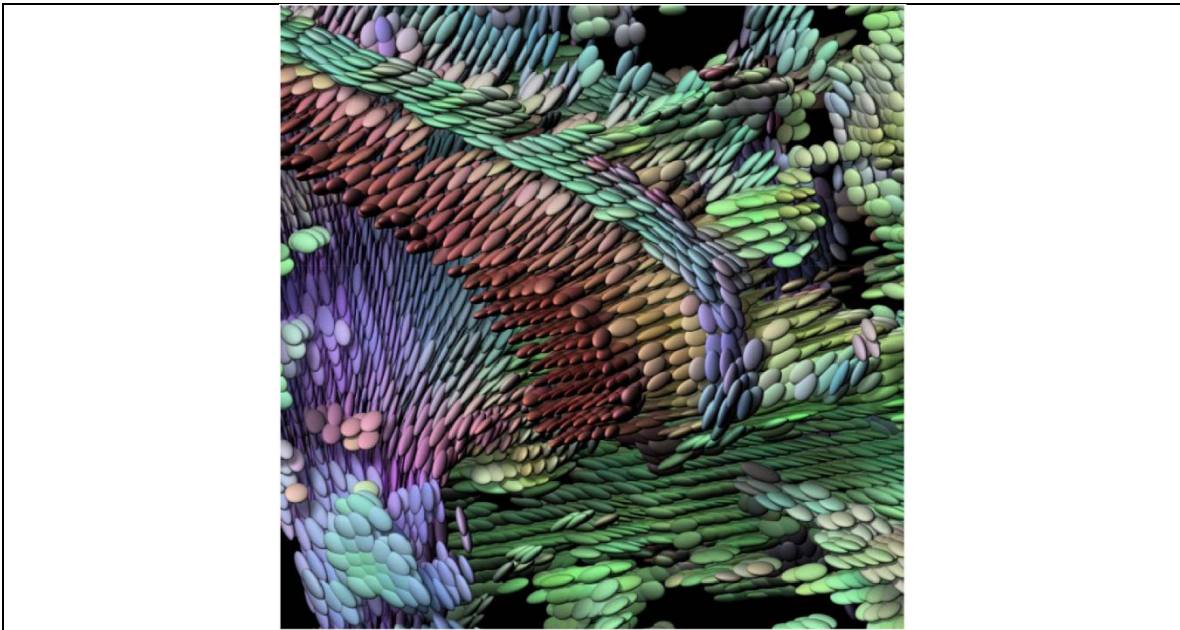
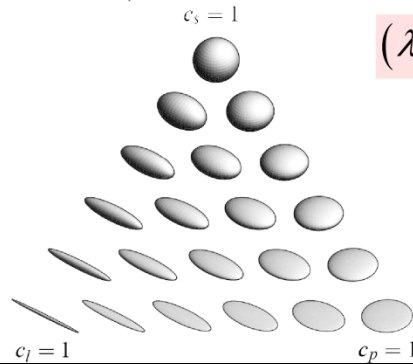
Anisotropy measure:

$$c_l = (\lambda_1 - \lambda_2) / (\lambda_1 + \lambda_2 + \lambda_3)$$

$$c_p = 2(\lambda_2 - \lambda_3) / (\lambda_1 + \lambda_2 + \lambda_3)$$

$$c_s = 3\lambda_3 / (\lambda_1 + \lambda_2 + \lambda_3)$$

$$(\lambda_1 \geq \lambda_2 \geq \lambda_3)$$



## 2. Vector Multiplication

### 2.1 Dot Product (Scalar Product)

If we have two vectors:

$$\vec{u}(x, y, z) = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$$

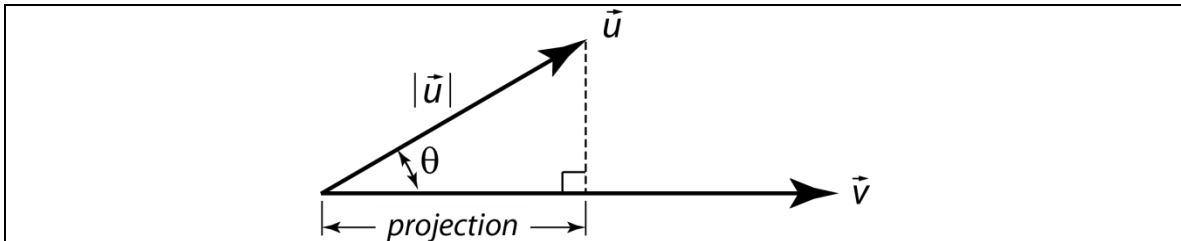
$$\vec{v}(x, y, z) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

Then, the dot product is defined as:

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

This is sometimes referred to as the scalar product, as the end result is a scalar and not a vector.

Physically if we have two vectors  $\vec{u}$  and  $\vec{v}$  with the angle  $\theta$  between them:



If we project  $\vec{u}$  down onto  $\vec{v}$  we get:

$$\cos \theta = \frac{\text{projection}}{|\vec{u}|}$$

$$\Rightarrow \text{projection} = |\vec{u}| \cos \theta$$

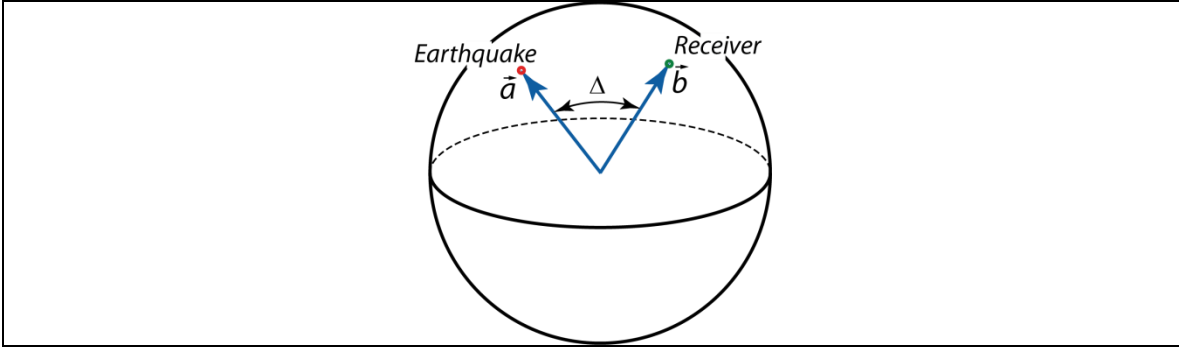
Physically, the dot product is the product of the projection of  $\vec{u}$  with the length of  $\vec{v}$ . That is,

$$\vec{u} \cdot \vec{v} = |\text{projection } \vec{u}| |\vec{v}|$$

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$$

This has many uses in geometry and geophysics.

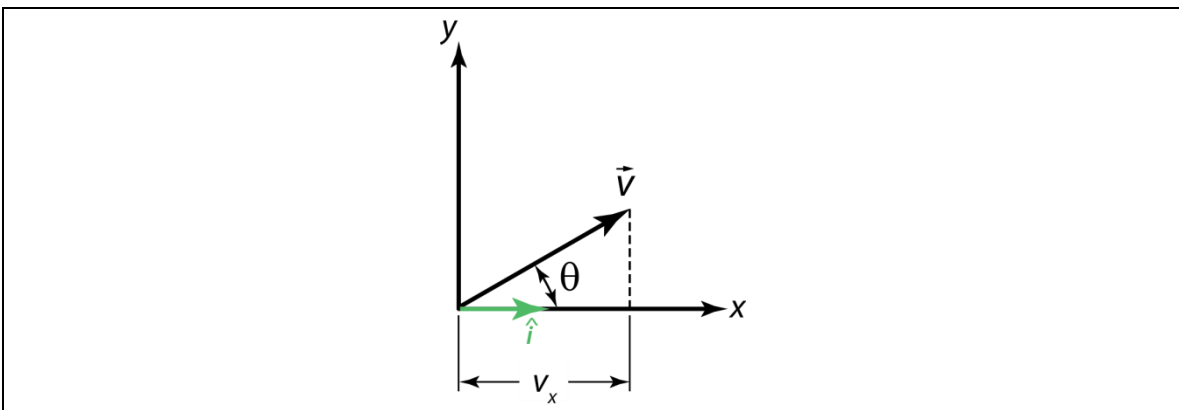
For example, what if you want to know the angular distance between two points on the surface of the Earth? An example could be distance between an earthquake and a receiver (epicentral distance) which is typically measured in degrees. One can simply define a vector from the center of the Earth to the Earthquake and another vector from the center of the Earth to the receiver:



Then we can just solve for the epicentral distance  $\Delta$ .

$$\cos \Delta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

Another use is to find the length of one vector quantity in the direction of another. For example, if we have a velocity vector  $\vec{v}$  as shown below:



We can find the x-component of the vector by taking the dot product with the  $\hat{i}$  vector:

$$\vec{v} \cdot \hat{i} = |\vec{v}| |\hat{i}| \cos \theta$$

$$\Rightarrow \vec{v} \cdot \hat{i} = |\vec{v}| \cos \theta$$

Which we see from simple trigonometry is the same as:

$$\cos \theta = \frac{v_x}{|\vec{v}|}$$

$$\Rightarrow v_x = |\vec{v}| \cos \theta$$



## 2.2 Cross Product (Vector Product)

Let us again take two vectors:

$$\vec{u}(x, y, z) = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$$

$$\vec{v}(x, y, z) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

Then, the cross product is defined as:

$$\vec{u} \times \vec{v} = [u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1]$$

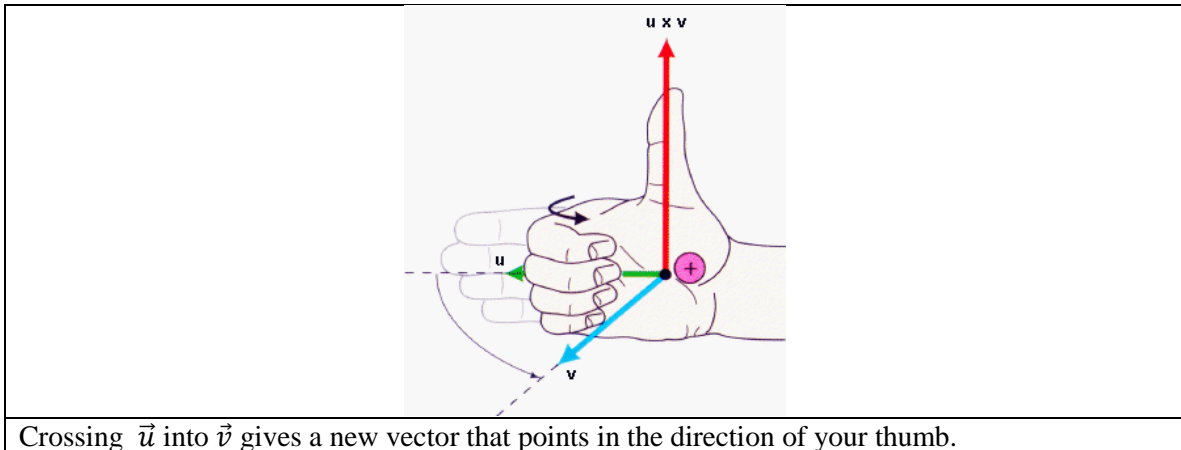
Note that the result of this multiplication is another vector that is perpendicular to the plane spanned by  $\vec{u}$  and  $\vec{v}$ .

Instead of memorizing the above formula, it is usually easier to remember how to do the cross product by using determinant notation:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\Rightarrow \vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

Recall that the direction the new vector points in is determined by the right hand rule:



Note that the length of the new vector is given as:

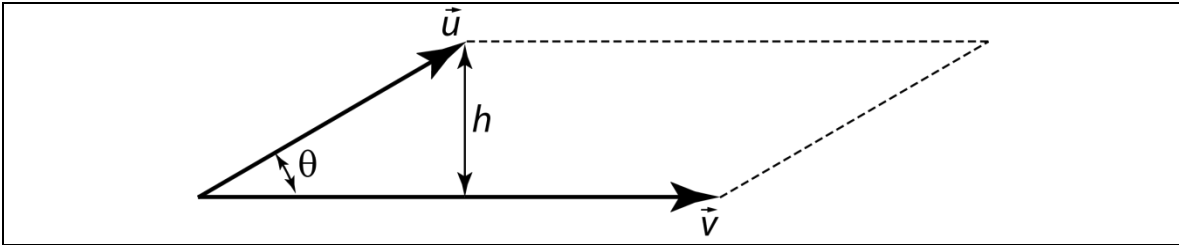
$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta$$

$$\text{If } \theta = 0^\circ \Rightarrow |\vec{u} \times \vec{v}| = 0.$$

$$\text{If } \theta = 90^\circ \Rightarrow |\vec{u} \times \vec{v}| = \text{maximum value.}$$

Thus, the cross product is a measure of how orthogonal two vectors are.

Physically, the cross product is the area of a parallelogram.



Note that the area of the above parallelogram is:  $Area = |\vec{v}|h$

$$\text{But, } \sin \theta = \frac{h}{|\vec{u}|} \Rightarrow h = |\vec{u}| \sin \theta$$

$$\text{And thus, } Area = |\vec{v}|h = |\vec{v}||\vec{u}| \sin \theta = \vec{u} \times \vec{v}$$

Some example uses of the cross product are as follows:

### 2.2.1 Linear velocity

Recall that linear velocity is defined as:

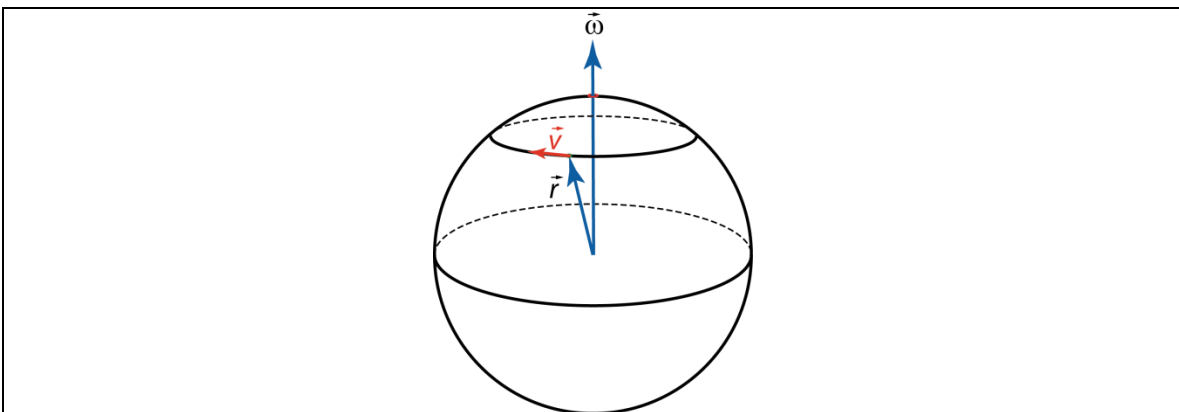
$$\vec{v} = \vec{\omega} \times \vec{r}$$

Where,

$\vec{\omega}$  = angular velocity vector

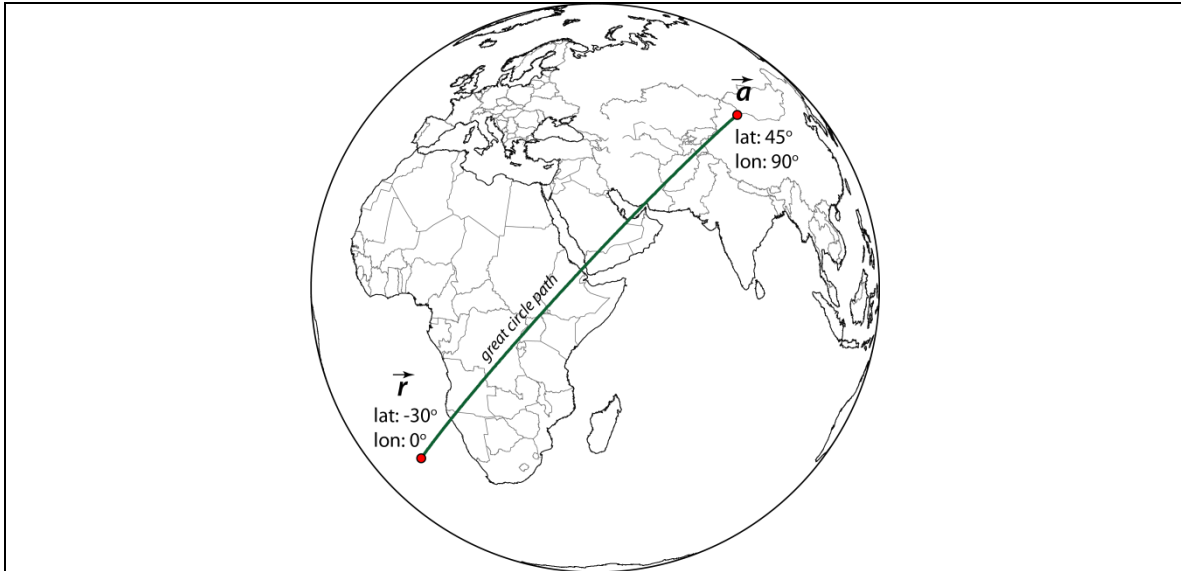
$\vec{r}$  = position vector

The linear velocity vector is then the vector  $\vec{v}$  that is perpendicular to  $\vec{\omega}$  and  $\vec{r}$  as shown in the diagram below.



**2.2.2 Great Circle Paths**

Another thing I like to use the cross product for is in tracing out great circle paths. Let's take a concrete example and work through it. Let's assume I have two points on the surface of the Earth as follows:



The first point has a latitude =  $-30^\circ$  and a longitude =  $0^\circ$ .

We can represent this as a position vector with its origin at the center of the Earth and the tip of the vector at latitude =  $-30^\circ$  and a longitude =  $0^\circ$ . Here, we just note that position on the surface of the Earth is related to spherical coordinate system as follows:

	<p><math>\lambda = \text{latitude} = 90^\circ - \beta</math></p> <p><math>r</math> = the magnitude of our vector <math>r</math>.</p> <p>Using the angle definitions to the left, we can write our point <math>r</math> as:</p> <p><math>r_x = r \sin \beta \cos \gamma</math></p> <p><math>r_y = r \sin \beta \sin \gamma</math></p> <p><math>r_z = r \cos \beta</math></p>
--	---

If we define:

- $\lambda_r = \text{latitude at point } r$
- $\varphi_r = \text{longitude at point } r$

Then, we can define the point  $r$ , in terms of the following vector components:

$$\vec{r} = [r \sin \beta \cos \gamma, r \sin \beta \sin \gamma, r \cos \beta]$$

And our point  $\vec{r}$  is then:

$$\vec{r} = [5517.4, 0, -3185.5]$$

Similarly, we can make a vector for our point  $\vec{a}$ .

$$\vec{a} = [0, 4505, 4505]$$

Now that I have vectors for  $\vec{r}$  and  $\vec{a}$ , I next determine a unit vector,  $\vec{u}$ , that is perpendicular to these position vectors. That is, I take the cross product between  $\vec{r}$  and  $\vec{a}$ . This gives me a new vector,  $\vec{u}$ , which I normalize to a unit vector  $\hat{u}$ , that can be used as an Euler pole in an Euler rotation matrix.

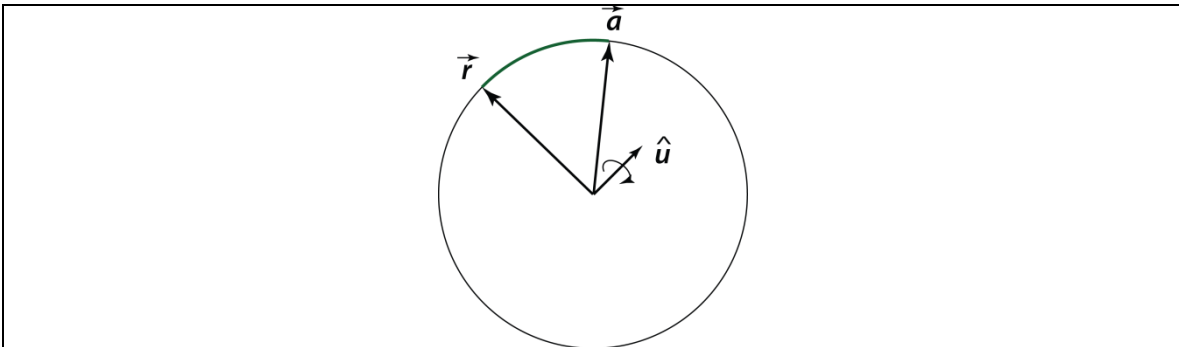
$$\vec{u} = \vec{r} \times \vec{a}$$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$$

For the above example,

$$\hat{u} = [0.3779, -0.6546, 0.6546]$$

Graphically, we get three vectors something like this:



Given the unit vector Euler pole as:

$$\vec{u} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k}$$

And initial position vector:

$$\vec{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$$

Then a generalized rotation is described as:

$$R = \begin{bmatrix} \cos \theta + u_x^2(1 - \cos \theta) & u_x u_y(1 - \cos \theta) - u_z \sin \theta & u_x u_z(1 - \cos \theta) + u_y \sin \theta \\ u_x u_y(1 - \cos \theta) + u_z \sin \theta & \cos \theta + u_y^2(1 - \cos \theta) & u_y u_z(1 - \cos \theta) - u_x \sin \theta \\ u_x u_z(1 - \cos \theta) - u_y \sin \theta & u_y u_z(1 - \cos \theta) + u_x \sin \theta & \cos \theta + u_z^2(1 - \cos \theta) \end{bmatrix}$$

The new position of the initial vector  $\vec{r}$  is given as:

$$\vec{r}_{new} = \left( r_x * R(1,1) + r_y * R(1,2) + r_z * R(1,3) \right) \hat{i} + \left( r_x * R(2,1) + r_y * R(2,2) + r_z * R(2,3) \right) \hat{j} \\ + \left( r_x * R(3,1) + r_y * R(3,2) + r_z * R(3,3) \right) \hat{k}$$

To trace out the great circle path, we just iterate in small angular steps where we update  $\vec{r}_{new}$  at each step.

### 3. Vector Operators

#### 3.1 Gradient

The gradient of a scalar function,  $f(x,y,z)$ , is defined as:

$$\vec{\nabla}(f) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

The gradient operator acts on a scalar field, but returns a vector field. The vector at each point in space points “uphill” in the direction of fastest increase of the function. The slope is given by the magnitude of the gradient vector.

As an example, let’s take the scalar function we used previously:

$$f(x, y) = (x^2 + 3y^2)e^{1-(x^2+y^2)}$$

Since this is in 2D we only need to compute:

$$\vec{\nabla}(f) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}$$

So, let’s calculate the derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} [(x^2 + 3y^2)e^{1-(x^2+y^2)}] \\ \Rightarrow \frac{\partial f}{\partial x} = (x^2 + 3y^2) \frac{\partial}{\partial x} [e^{1-(x^2+y^2)}] + e^{1-(x^2+y^2)} \frac{\partial}{\partial x} [(x^2 + 3y^2)] \\ \Rightarrow \frac{\partial f}{\partial x} = (x^2 + 3y^2)(-2x)e^{1-(x^2+y^2)} + (2x)e^{1-(x^2+y^2)}$$

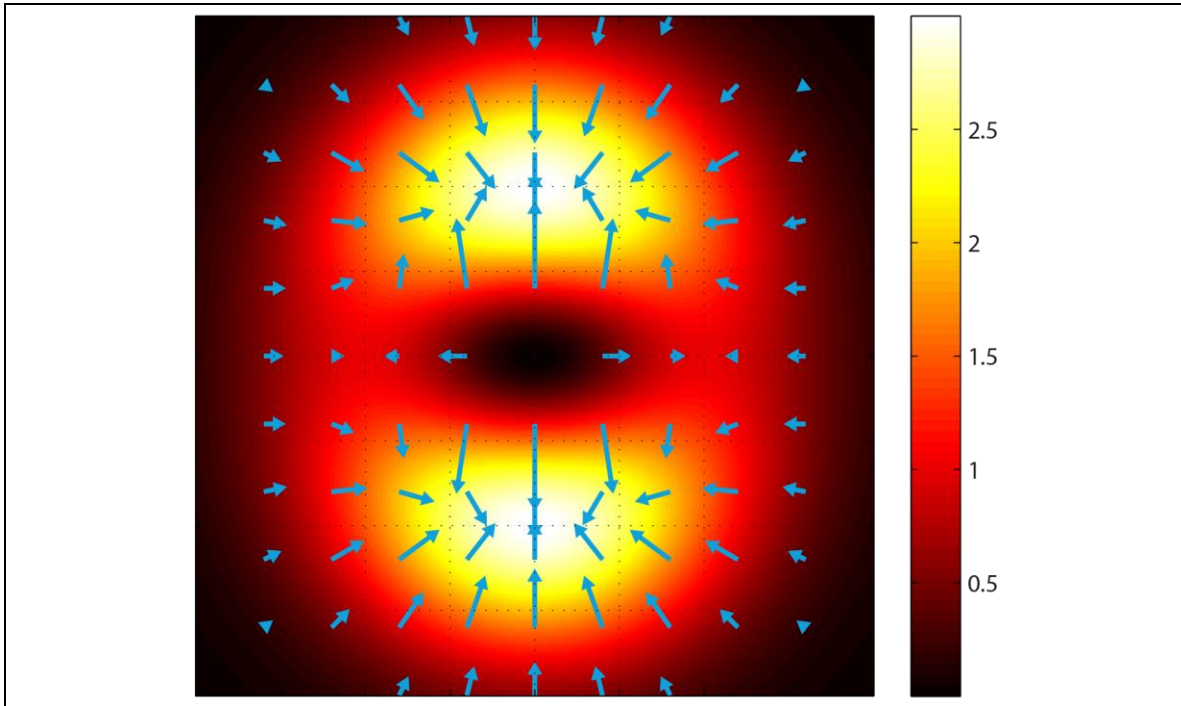
Similarly,

$$\frac{\partial f}{\partial y} = (x^2 + 3y^2)(-2y)e^{1-(x^2+y^2)} + (6y)e^{1-(x^2+y^2)}$$

Which are the two components of our gradient vector. That is:

$$\vec{\nabla}(f) = [(x^2 + 3y^2)(-2x)e^{1-(x^2+y^2)} + (2x)e^{1-(x^2+y^2)}]\hat{i} + [(x^2 + 3y^2)(-2y)e^{1-(x^2+y^2)} + (6y)e^{1-(x^2+y^2)}]\hat{j}$$

Let's plot this vector field on top of the plain color version of our scalar field from above. Here it is obvious the vectors point “uphill” and the largest vector magnitudes are where the elevation change is the most rapid.



The gradient operator is enormously important in geophysical fields. This is because in geophysics we deal with *potential fields*. Examples are gravitational potential, magnetic potential, etc. These fields are all scalar fields.

For example, let's consider gravitational potential. Gravitational potential is related to gravitational potential energy, but it is normalized such that it is in reference to a 1-kg test mass. That is, gravitational potential is the gravitational potential energy a 1-kg mass has in the presence of another mass. This potential energy can be described with a single number for each point in space (and thus is often used as it is cheaper to store 1 number, than say 3 numbers for a vector).

For now let's just assume the Earth is a sphere. In this case its gravitational potential is given by:

$$U(\vec{r}) = -\frac{GM_E}{|\vec{r}|}$$

In later lectures we will deal with a better approximation for the Earth's gravitational potential, as the Earth is not spherical, rather shaped like an oblate spheroid, but for now this will work. Now, let's consider adding in the Moon. We can just use superposition to get the gravitational potential of the Earth-Moon system as:

$$U(\vec{r}) = -\frac{GM_E}{|\vec{r} - \vec{r}_{CE}|} - \frac{GM_M}{|\vec{r} - \vec{r}_{CM}|}$$

Where,

$M_E$  - Is the mass of the Earth.

$M_M$  - Is the mass of the Moon.

$\vec{r}_{CE}$  - Is the position vector to the center of the Earth.

$\vec{r}_{CM}$  - Is the position vector to the center of the Moon.

So, we can code up the potential in MATLAB:

```
clear all
ME = 5.972e24;      % Mass of the Earth (kg)
MM = 7.348e22;      % Mass of the Moon (kg)
G = 6.67384e-11;   % Big G, (m^3 kg^-1 s^-2)
Rm = 384000e3;      % Distance to the Moon (m)

Xm = Rm;
Ym = 0.;

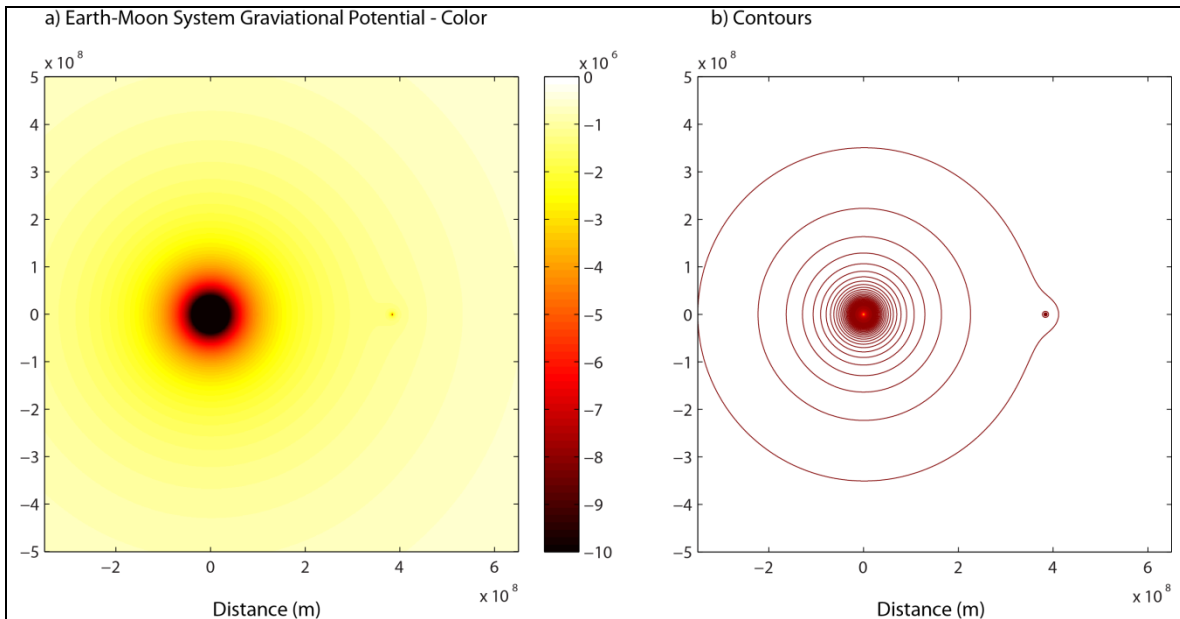
xgrid = linspace(-350000e3, 650000e3, 1000);
ygrid = linspace(-500000e3, 500000e3, 1000);
[x,y]=meshgrid(xgrid,ygrid);

rmag = sqrt(x.^2 + y.^2);
U1 = G*ME./rmag;
Ax = x - Xm;
Ay = y - Ym;
r2mag = sqrt(Ax.^2 + Ay.^2);
U2 = G*MM./r2mag;
U = -U1-U2;

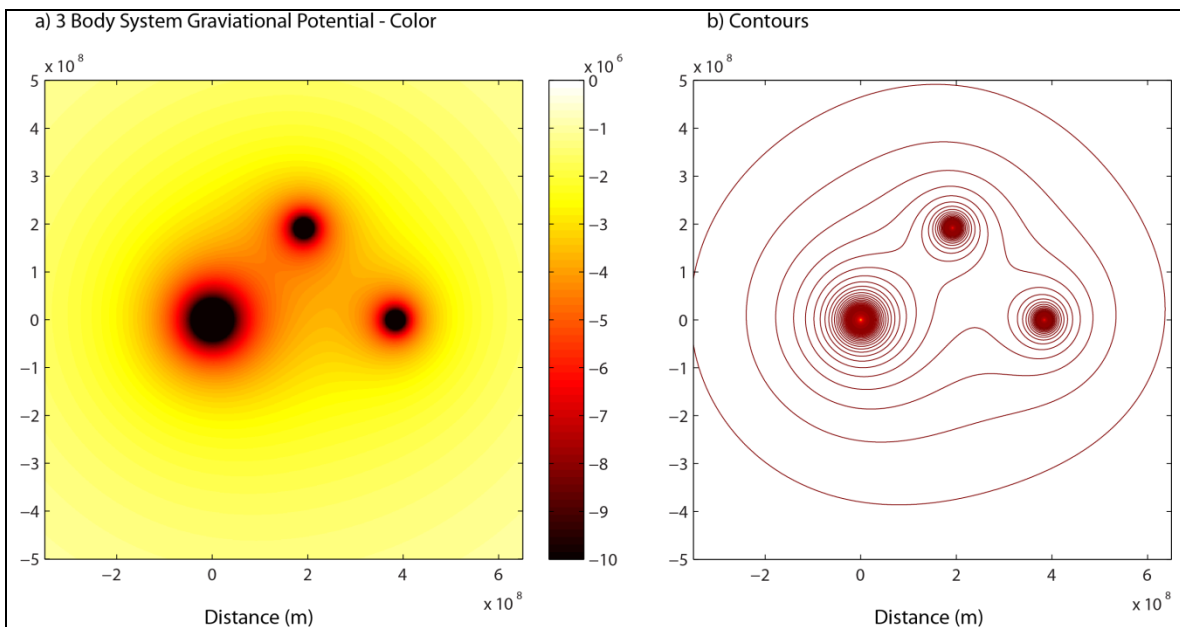
figure(1)
colormap(hot)
set(gcf, 'PaperOrientation', 'landscape', 'PaperPosition', ...
    [.25 .25 10.5 8], 'PaperType', 'usletter');
imagesc(xgrid,ygrid,U)
caxis([-1e7 0.0])
set(gca, 'YDir', 'normal')
axis square
colorbar

%figure 2 - plot contour lines
figure(2)
set(gca, 'YDir', 'normal')
[C,h]=contour(xgrid,ygrid,U,1000);
axis square
```

So, just plotting this up we get:



In the above figure the Moons effect is small, so to make something more outrageous let's not model the exact Earth/Moon system, but add a couple of moons with  $\frac{1}{2}$  the mass of the Earth:



Now this is getting a little more interesting. If we stick a mass somewhere in the system it gets a little trickier to see where it would go. Here, let me point out the value of the contours. The contours are “equipotential” surfaces. That is, areas of equal potential energy. The gravitational field always points perpendicularly to the equipotential lines, so the contour plot may actually help us out a bit. But, better yet would be to plot the actual gravity field as a vector field. So, how do we do that? We are in luck, because we can always go from the scalar potential field to the vector field through the following relation:



$$\vec{g} = -\vec{\nabla}U$$

That is, we can get the full vector gravitational force field by taking the negative of the gradient of the potential field. In general the relationships between scalar fields and vector fields in geophysics are always through the gradient operator.

Recall that the gravitational force is described as:

$$\vec{F}(\vec{r}) = -\frac{GMm}{r^2}\hat{r}$$

And the gravitational field is just the gravitational force normalized per unit mass:

$$\vec{g}(\vec{r}) = \frac{\vec{F}(\vec{r})}{m}$$

OK, but how do we do this now since we don't have equations to calculate the derivatives for the gradient from? Well, we can just make a simple approximation to the derivatives using a finite difference operator.

For example, we can make an approximation of the derivative at each grid point in the x-direction as:

$$\frac{\partial f}{\partial x} \approx \frac{f(x+h) - f(x-h)}{2h}$$

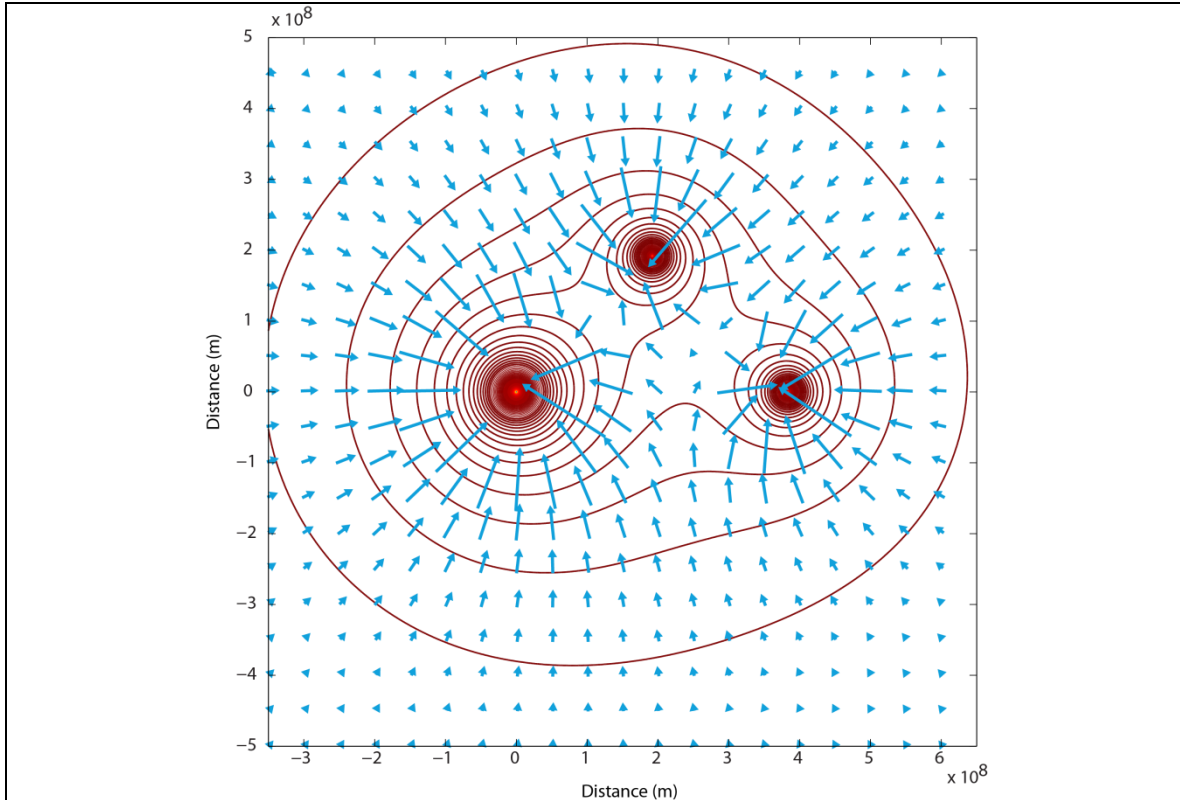
Where, h is the grid spacing.

Thus, we can estimate the gravitational force field as:

$$\vec{g} = -\vec{\nabla}(U) \approx -\left(\frac{U(x+dx) - U(x-dx)}{2dx}\right)\hat{i} - \left(\frac{U(y+dy) - U(y-dy)}{2dy}\right)\hat{j}$$

Where we let  $dx$  and  $dy$  be the grid spacing in the  $x$ - and  $y$ - direction respectively.

Let's code this up for the 3-body situation plotted above, and plot the gravitational field vectors on top of the gravitational potential contours.



It is important to note the field vectors are perpendicular to the equipotential contours, pointing in the direction a mass would be attracted at each point, and of course the length of the vectors describe the strength of the gravitational force at each point. I actually deleted some of the vectors near the centers of the planets because the vectors were huge in these locations and cluttered up the view.

### 3.2 Divergence

The input to the divergence operator is a vector field and the output is a scalar field. The divergence is defined as follows:

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

What the divergence physically represents can be understood if we make an analogy that the vector field represents a gas or a fluid. For example, if the vectors represent velocity of a fluid flowing, then the divergence measures compression or decompression of the fluid in an infinitesimally small volume around the point of interest. Specifically the divergence of  $F$  represents the rate of expansion per unit volume under the flow of the gas or fluid.

Wikipedia makes a nice description. Let's assume our vector field is the velocity of air movement inside a room. In some places this air may be heated. This would cause the air to expand or decompress and the divergence of the vector field in this region would be positive. If on the other hand, the air were cooled in some region of the room the air would contract or decompress and the divergence would have a negative value. A zero-valued divergence means that in this region the air is neither expanding nor contracting.

As an example, let's let:

$$\vec{F}(x, y) = x\hat{i} + y\hat{j}$$

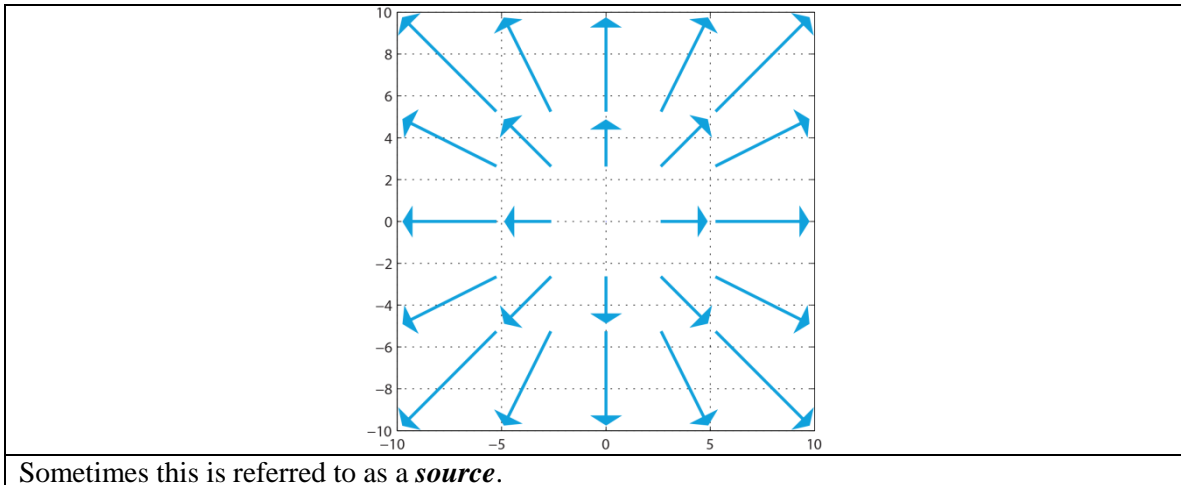
We can calculate the divergence as:

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)$$

$$\Rightarrow \vec{\nabla} \cdot \vec{F} = 1 + 1 = 2$$

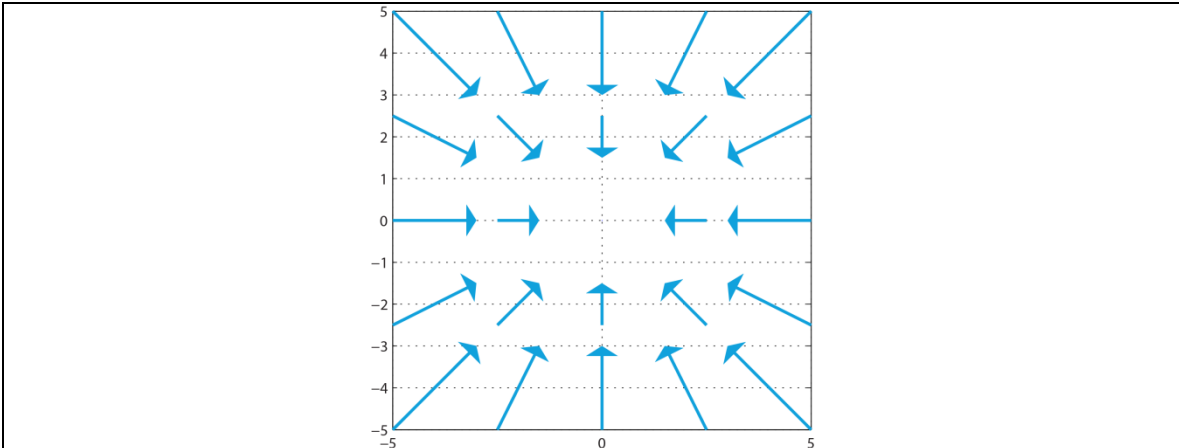
Since this is just a positive constant, we expect the vector field is expanding everywhere. Let's plot the vector field:



As another example let:

$$\vec{F}(x,y) = -x\hat{i} - y\hat{j}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{F} = -1 - 1 = -2$$



Similar to above, this is referred to as a *sink*.

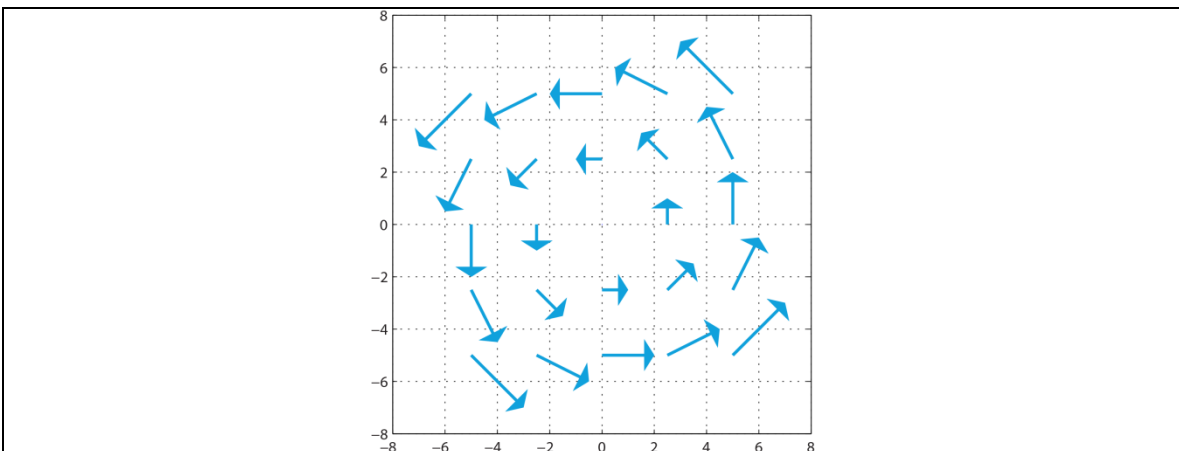
In considering the above two examples, people sometimes refer to the divergence as a measure of *sources* and *sinks* (e.g., a heat source or heat sink is an applicable analogy).

Another example:

$$\vec{F}(x,y) = -y\hat{i} + x\hat{j}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)$$

$$\Rightarrow \vec{\nabla} \cdot \vec{F} = 0 + 0 = 0$$



This example has zero divergence. That is, there is nowhere that is undergoing expansion or compression. Often if a field doesn't undergo expansion or compression it is termed *incompressible*.

### 3.2.1 P-wave propagation

As a seismologically relevant example, let's consider the propagation of seismic P-waves. Section 3.4.1 of Peter Shearer's textbook on Seismology shows that the ground displacement,  $\vec{u}$ , of a 1 Hz harmonic plane P-wave propagating in the positive  $x$ -direction can be described as follows:

$$\vec{u}_x(x, y, t) = a \cos \left[ 2\pi \left( t - \frac{x}{6} \right) \right]$$

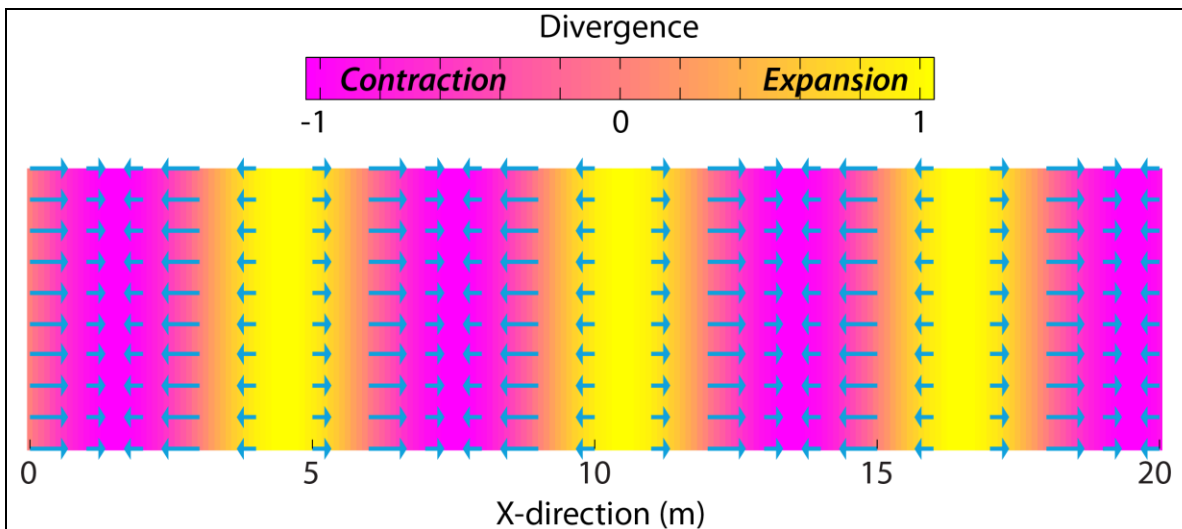
$$\vec{u}_y(x, y, t) = 0$$

Here,  $\vec{u}_x$  and  $\vec{u}_y$  are the vector components of the displacement. The amplitude of the seismic wave is given by  $a$ .

We can calculate the divergence of  $\vec{u}$  as follows:

$$\begin{aligned} \vec{\nabla} \cdot \vec{u} &= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \\ \Rightarrow \vec{\nabla} \cdot \vec{u} &= \frac{\partial}{\partial x} \left( a \cos \left[ 2\pi \left( t - \frac{x}{6} \right) \right] \right) + \frac{\partial}{\partial y} (0) \\ \Rightarrow \vec{\nabla} \cdot \vec{u} &= \frac{\pi a}{3} \sin \left[ 2\pi \left( t - \frac{x}{6} \right) \right] \end{aligned}$$

The following plot shows the vector field (displacement, blue arrows) overlain on top of the scalar field (divergence, colored background) at a single instance in time.



Since this is time varying, try out the following MATLAB script and watch the P-wave travel. Here we can immediately see that the P-wave propagates by successive expansion (positive divergence) and compression (negative divergence) of the ground.

```

clear all

% set up some initial parameters
%-----%
xmin = 0;      %min x-axis
xmax = 20;     %max x-axis
nx  = 21;     %number of vectors in x-direction
ymin = 0;     %min y-axis
ymax = 10;    %max y-axis
ny  = 31;    %number of vectors in y-direction
a   = 1.0;   %amplitude
Vp  = 6.0;   %P-wave velocity (km/s)
%-----%

for t=0:0.01:5 %begin loop through time

% calculate the vector field for the P-wave displacement
%-----%
xgrid = linspace(xmin,xmax,nx);
ygrid = linspace(ymin,ymax,ny);
[Xmesh,Ymesh]=meshgrid(xgrid,ygrid);

for i=1:length(xgrid)
    for j=1:length(ygrid)
        x = Xmesh(j,i);
        y = Ymesh(j,i);
        Vx(j,i) = a*cos(2*pi*(t-x/Vp));
        Vy(j,i) = 0;
    end
end
%-----%

% calculate the scalar potential - divergence of u
%-----%
x2grid = linspace(xmin,xmax,200);
y2grid = linspace(ymin,ymax,200);
[x,y]=meshgrid(x2grid,y2grid);
for i=1:length(x2grid)
    for j=1:length(y2grid)
        xx = x(i,j);
        divu(i,j) = -sin(2*pi*(t-xx/Vp))*(-2*pi/Vp);
    end
end
%-----%

% plot the vector field over the scalar field
%-----%
figure(1)
colormap(spring)
set(gcf,'PaperOrientation','landscape','PaperPosition', ...
    [.25 .25 10.5 8],'PaperType','usletter');

```

```

imagesc(x2grid,y2grid,divu)
h = colorbar;
h.Label.String = 'Divergence';
set(gca, 'YDir', 'normal')
hold on
quiver(Xmesh,Ymesh,Vx,Vy,0.5,'filled','b','LineWidth',1)
axis([xmin xmax ymin ymax])
xlabel('X-direction')
strtit=strcat('Time = ',num2str(t),' (sec)');
title(strtit)
grid on
hold off
pause
end %end loop in time

```

### 3.3 Curl

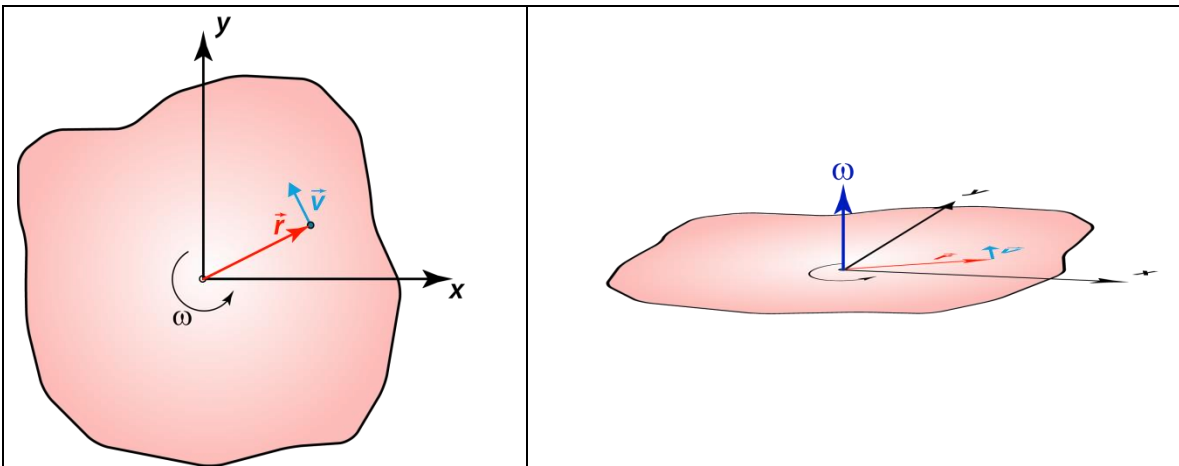
The curl of a vector field is defined as follows:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\Rightarrow \vec{\nabla} \times \vec{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} - \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$$

The curl of a vector field also returns a vector field. It is important to note that the curl is only defined in 3-dimension.

To understand what the curl physically represents let's consider what happens for a body that is undergoing rigid rotation. In the drawing below the red shaped object is a single body that is rotating around the z-axis (out of the plane of the page). It is rotating with angular velocity:  $\omega$ .



At any position, defined by the position vector,  $\vec{r}$ , we can determine the linear velocity,  $\vec{v}$ , as described above in section 2.2.1.

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Where we define the vector  $\vec{\omega}$  as pointing in the direction of the z-axis, with a magnitude of the angular velocity,  $\omega$ .

Thus, for the two vectors:

$$\vec{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$$

Since, we define our position vector with respect to the coordinate origins:

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

And,

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

Since, the angular velocity vector just points in the direction of the positive z-axis this implies that:

$$\Rightarrow \vec{\omega} = 0 \hat{i} + 0 \hat{j} + \omega \hat{k}$$

Hence,

$$\vec{v} = \vec{\omega} \times \vec{r}$$

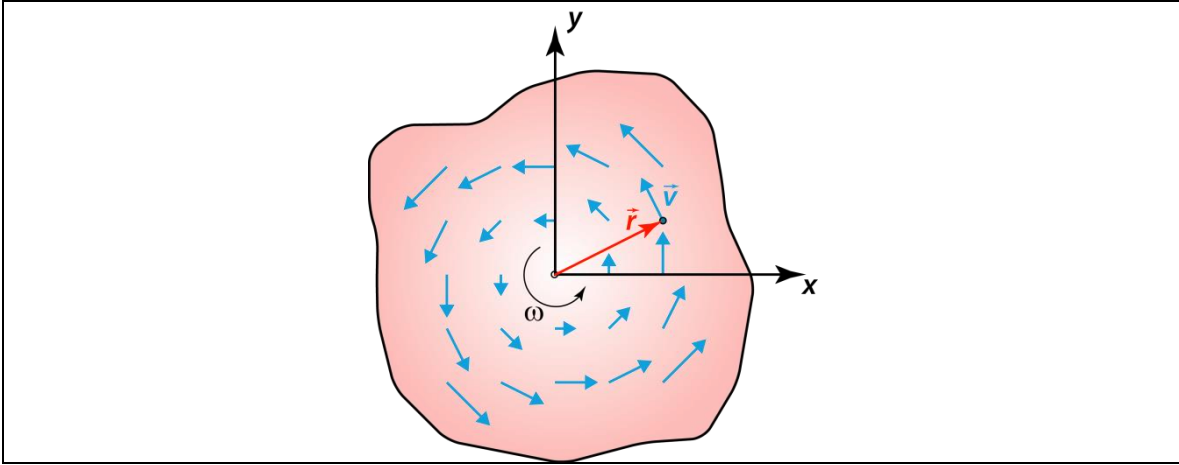
$$\Rightarrow \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix}$$

$$\Rightarrow \vec{v} = (-\omega y) \hat{i} - (-\omega x) \hat{j} + (0) \hat{k}$$

$$\Rightarrow \vec{v} = (-\omega y) \hat{i} + (\omega x) \hat{j}$$

This is a familiar vector field which looks as follows:



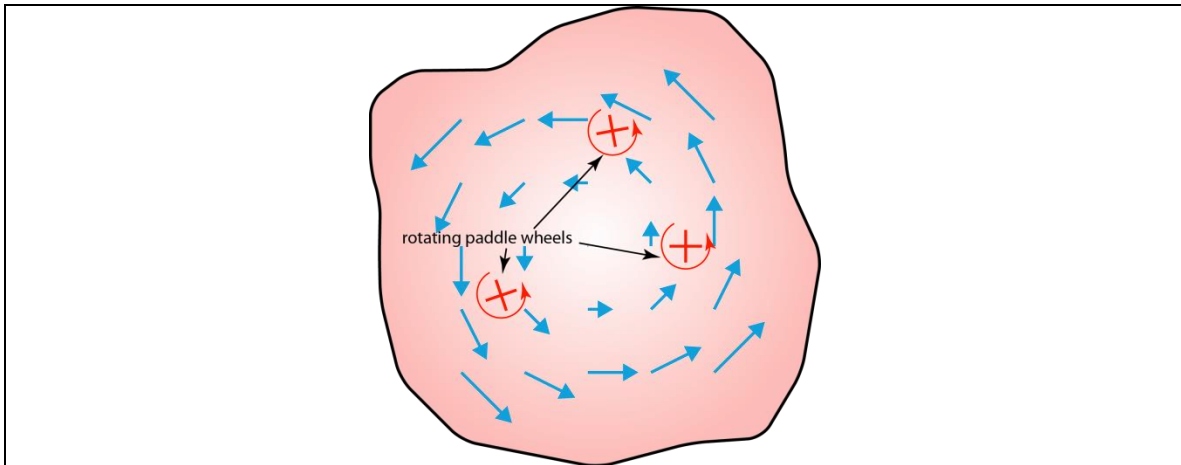


So, let's calculate the curl of this vector field:

$$\begin{aligned}\vec{\nabla} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ \Rightarrow \vec{\nabla} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} \\ \Rightarrow \vec{\nabla} \times \vec{v} &= \frac{\partial}{\partial z}(\omega x)\hat{i} - \frac{\partial}{\partial z}(-\omega y)\hat{j} + \left[ \frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y) \right] \hat{k} \\ \Rightarrow \vec{\nabla} \times \vec{v} &= 0\hat{i} + 0\hat{j} + \left[ \frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y) \right] \hat{k} \\ \Rightarrow \vec{\nabla} \times \vec{v} &= 0\hat{i} + 0\hat{j} + [\omega + \omega]\hat{k} \\ \Rightarrow \vec{\nabla} \times \vec{v} &= 0\hat{i} + 0\hat{j} + 2\omega\hat{k}\end{aligned}$$

So, for the rotation of a rigid body, the curl of the velocity vector field is a new vector field whose magnitude is the same at each point and the vector points in the direction along the axis of rotation. Note that the magnitude of the new vector field is twice that of the angular speed. Thus, the physical interpretation of the curl is that: In an infinitesimal area surrounding a point in our vector field, the curl provides us with a vector that tells us the direction and twice the magnitude of rotation of a rigid body that rotates as the vector field does near that point. If the curl of the vector field is zero, then the vector field is termed *irrotational*.

An intuitive way to look at the curl is to visualize the vector field as the flow of a fluid. Then, we stick a paddle wheel into the fluid and see if it would rotate. If it rotates, then it has a non-zero curl.



### 3.3.1 P-wave propagation

In our discussion on divergence we used the example of P-wave propagation.

$$\vec{u}_x(x, y, t) = a \cos \left[ 2\pi \left( t - \frac{x}{6} \right) \right]$$

$$\vec{u}_y(x, y, t) = 0$$

$$\vec{u}_z(x, y, t) = 0$$

What is the curl of this displacement vector field?

$$\vec{\nabla} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix}$$

$$\Rightarrow \vec{\nabla} \times \vec{u} = \left( \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) \hat{i} - \left( \frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z} \right) \hat{j} + \left( \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{k}$$

$$\Rightarrow \vec{\nabla} \times \vec{u} = (0 - 0)\hat{i} - (0 - 0)\hat{j} + (0 - 0)\hat{k}$$

$$\Rightarrow \vec{\nabla} \times \vec{u} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

The curl is zero everywhere. What this means is that there are no rigid-body rotations associated with the passage of a P-wave.

### 3.3.2 S-wave propagation

S-waves propagate by a shearing motion. Section 3.5 of Peter Shearer's Seismology textbook also provides equations for a harmonic S-wave propagating in the positive  $x$ -direction. We can once again write time-dependent equations for the propagation of a 1-Hz harmonic S-wave

traveling with a seismic wave speed of 6 km/s. The components of the displacement vector field are as follows:

$$\begin{aligned}\vec{u}_x(x, y, t) &= 0 \\ \vec{u}_y(x, y, t) &= -\frac{a\pi}{3} \sin\left[2\pi\left(t - \frac{x}{6}\right)\right] \\ \vec{u}_z(x, y, t) &= \frac{a\pi}{3} \sin\left[2\pi\left(t - \frac{x}{6}\right)\right]\end{aligned}$$

We can calculate the curl of the S-wave displacement:

$$\begin{aligned}\vec{\nabla} \times \vec{u} &= \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\hat{i} - \left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right)\hat{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{k} \\ \Rightarrow \vec{\nabla} \times \vec{u} &= (0 - 0)\hat{i} - \left(\frac{\partial u_z}{\partial x} - 0\right)\hat{j} + \left(\frac{\partial u_y}{\partial x} - 0\right)\hat{k} \\ \Rightarrow \vec{\nabla} \times \vec{u} &= 0\hat{i} + \left(\frac{a\pi^2}{9} \cos\left[2\pi\left(t - \frac{x}{6}\right)\right]\right)\hat{j} + \left(\frac{a\pi^2}{9} \cos\left[2\pi\left(t - \frac{x}{6}\right)\right]\right)\hat{k}\end{aligned}$$

OK, S-waves are a little bit more complicated than P-waves because motion is in the y-direction and z-direction. Typically, we split this up into SH-waves and SV-waves. For right now, let's just consider the SV-waves. That is, let's consider the y-direction as the up-down direction, and not consider the z-direction.

In this case, we just worry about the following displacements:

$$\begin{aligned}\vec{u}_x(x, y, t) &= 0 \\ \vec{u}_y(x, y, t) &= -\frac{a\pi}{3} \sin\left[2\pi\left(t - \frac{x}{6}\right)\right]\end{aligned}$$

To start out, let's visualize the vector field with a MATLAB script.

```
clear all

a = 1.0;
t = 0.0;

for t=0:0.01:5 %start loop through time

% calculate the vector field for the P-wave displacement
xgrid = linspace(0,20,21);
ygrid = linspace(0,10,11);
[X,Y]=meshgrid(xgrid,ygrid);

for i=1:length(xgrid)
    for j=1:length(ygrid)
        xx = X(j,i);
        yy = Y(j,i);
```

```

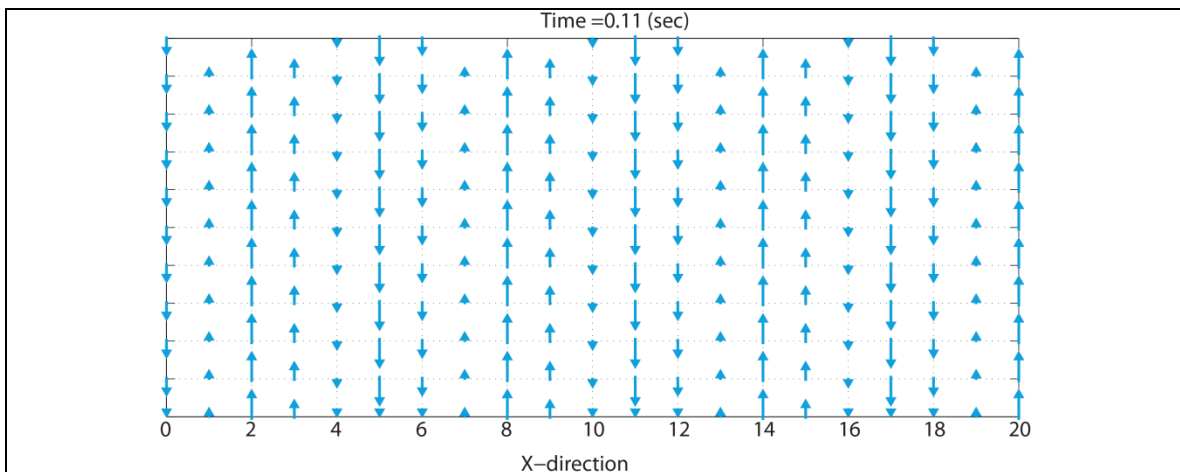
    Vx(j,i) = 0;
    Vy(j,i) = -(a*2*pi/6)*sin(2*pi*(t-xx/6));
end
end

% plot the vector field
figure(1)
quiver(X,Y,Vx,Vy,0.5,'filled','b','LineWidth',1,'MaxHeadSize',.4)
axis([0 20 0 10])
xlabel('X-direction')
stitle=strcat('Time = ',num2str(t),' (sec)');
title(stitle)
grid on
hold off
pause

end %end loop in time

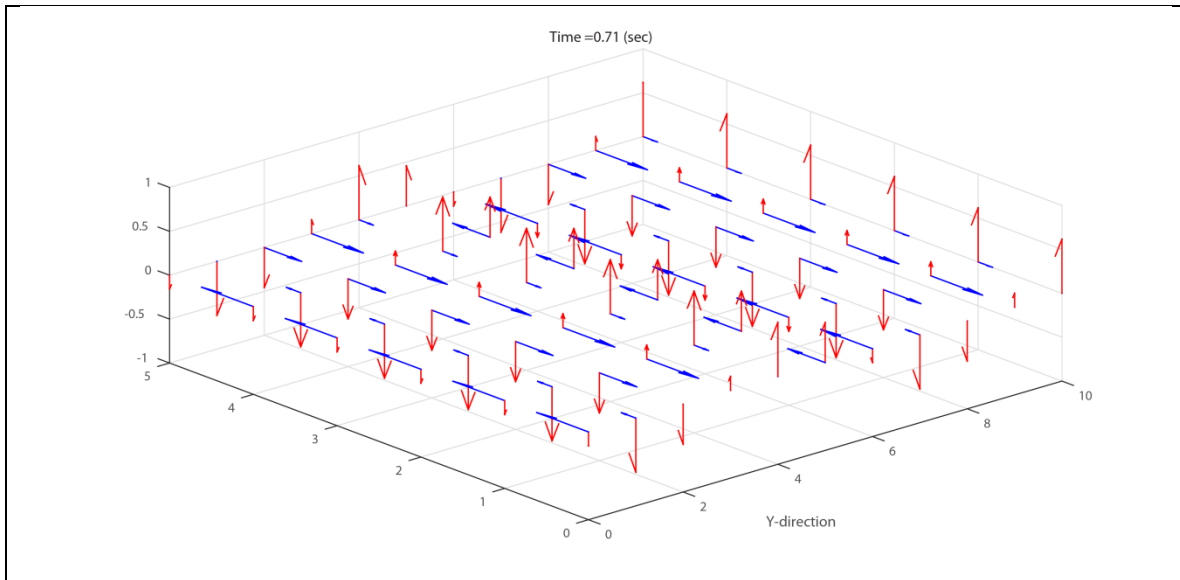
```

This results in a vector field that looks as follows:

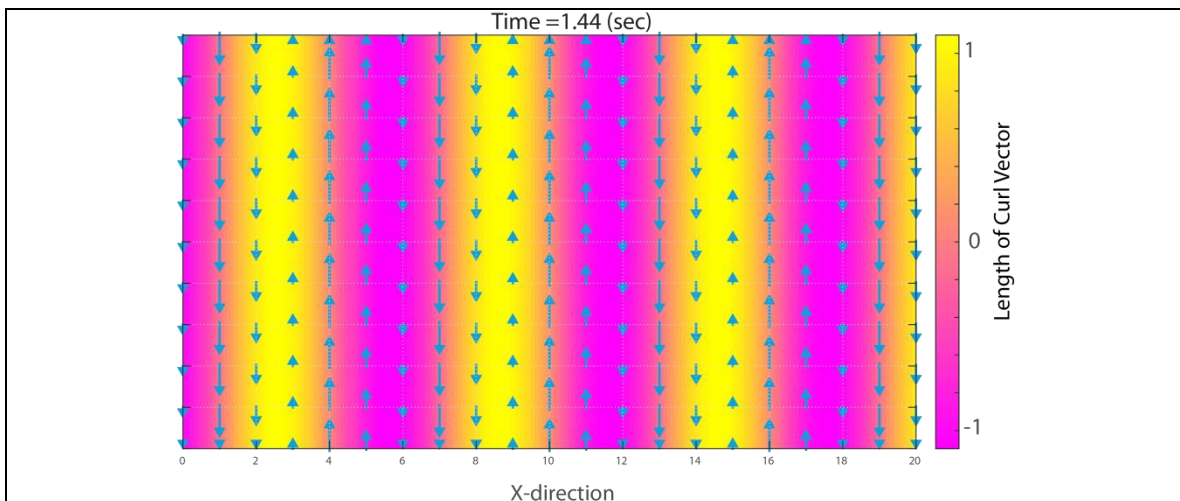


It may not be obvious from looking at the above vector field that it has a non-zero curl. But, I would suggest thinking about putting a paddle wheel in the vector field and imagining if it would get turned.

So, let's visualize the curl of this vector field. Truly, the curl gives us a new vector. This vector will be perpendicular to the page. So, it should be plotted in 3D. The following image gives us an idea. Here the blue vectors are drawn as in the previous figure, and the red vectors show us the curl.



The above image is a little bit more difficult to visualize. So, instead of drawing out the vectors for the curl, let's just plot their length as a color image and overlay the displacement field.



This makes visualization a little bit easier. Here we can see that the yellow regions have a positive curl (rotation vector out of the plane of the page – counterclockwise rotation) and the reddish regions have a negative curl (rotation vector into the plane of the page – clockwise rotation).

### 3.4 Laplacian

The Laplacian operator works on a scalar field and produces a scalar field. It is defined as:

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

For example,

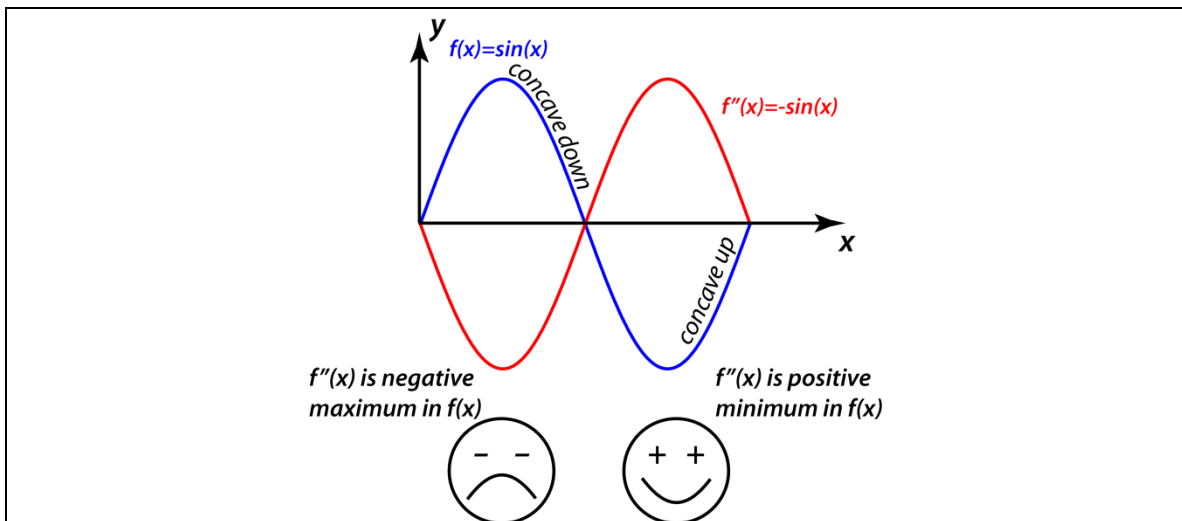
$$\begin{aligned}
 f(x, y, z) &= xy^2 + z^3 \\
 \Rightarrow \vec{\nabla}^2 f &= \frac{\partial^2}{\partial x^2}(xy^2 + z^3) + \frac{\partial^2}{\partial y^2}(xy^2 + z^3) + \frac{\partial^2}{\partial z^2}(xy^2 + z^3) \\
 \Rightarrow \vec{\nabla}^2 f &= \frac{\partial}{\partial x}(y^2 + 0) + \frac{\partial}{\partial y}(2xy + 0) + \frac{\partial}{\partial z}(0 + 3z^2) \\
 \Rightarrow \vec{\nabla}^2 f &= 0 + 2x + 6z \\
 \Rightarrow \vec{\nabla}^2 f &= 2x + 6z
 \end{aligned}$$

What does this physically represent?

Physically, the Laplacian measures the local concentration of  $f$ . That is, it measures the local excess value of  $f$  over its mean value.

First, let's consider the physical meaning of the 2<sup>nd</sup> derivative, which is a measure of concavity. Remember that we can tell if we have a minimum (concave up) or a maximum (concave down) by the 2<sup>nd</sup> derivative test. For example, in 1D consider:

$$\begin{aligned}
 f(x) &= \sin x \\
 \Rightarrow \frac{\partial^2 f(x)}{\partial x^2} &= -\sin x
 \end{aligned}$$



Let's consider the equation for the peaks we looked at in section 1.

$$f(x, y) = (x^2 + 3y^2)e^{1-(x^2+y^2)}$$

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Calculating the first derivatives:

$$\frac{\partial f}{\partial x} = (-2x)(x^2 + 3y^2)e^{1-(x^2+y^2)} + (2x)e^{1-(x^2+y^2)}$$

$$\frac{\partial f}{\partial y} = (-2y)(x^2 + 3y^2)e^{1-(x^2+y^2)} + (6y)e^{1-(x^2+y^2)}$$

And now calculating the 2<sup>nd</sup> derivatives:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= (-2x)(-2x)(x^2 + 3y^2)e^{1-(x^2+y^2)} + e^{1-(x^2+y^2)}(-6x^2 - 6y^2) \\ &\quad + (2x)e^{1-(x^2+y^2)}(-2x) + (2)e^{1-(x^2+y^2)} \end{aligned}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = (4x^4 + 12x^2y^2)e^{1-(x^2+y^2)} - (6x^2 + 6y^2)e^{1-(x^2+y^2)} - (4x^2)e^{1-(x^2+y^2)} + 2e^{1-(x^2+y^2)}$$

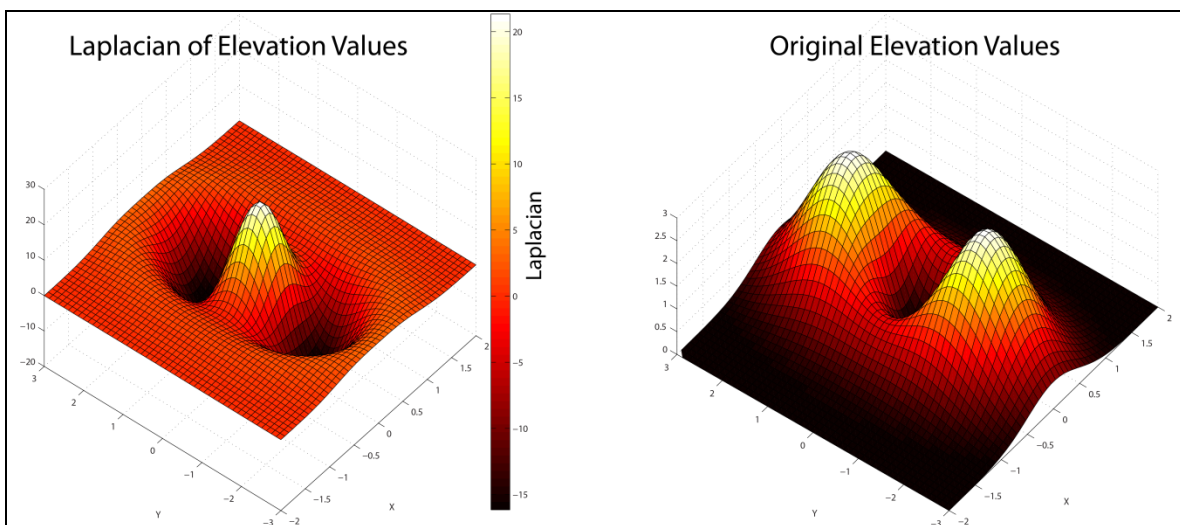
$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = e^{1-(x^2+y^2)}[4x^4 - 10x^2 + 12x^2y^2 - 6y^2 + 2]$$

And,

$$\frac{\partial^2 f}{\partial x^2} = e^{1-(x^2+y^2)}[-2x^2 - 30y^2 + 12y^4 + 4x^2y^2 + 6]$$

Thus,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{1-(x^2+y^2)}[4x^4 + 12y^4 - 12yx^2 - 36y^2 + 16x^2y^2 + 8]$$

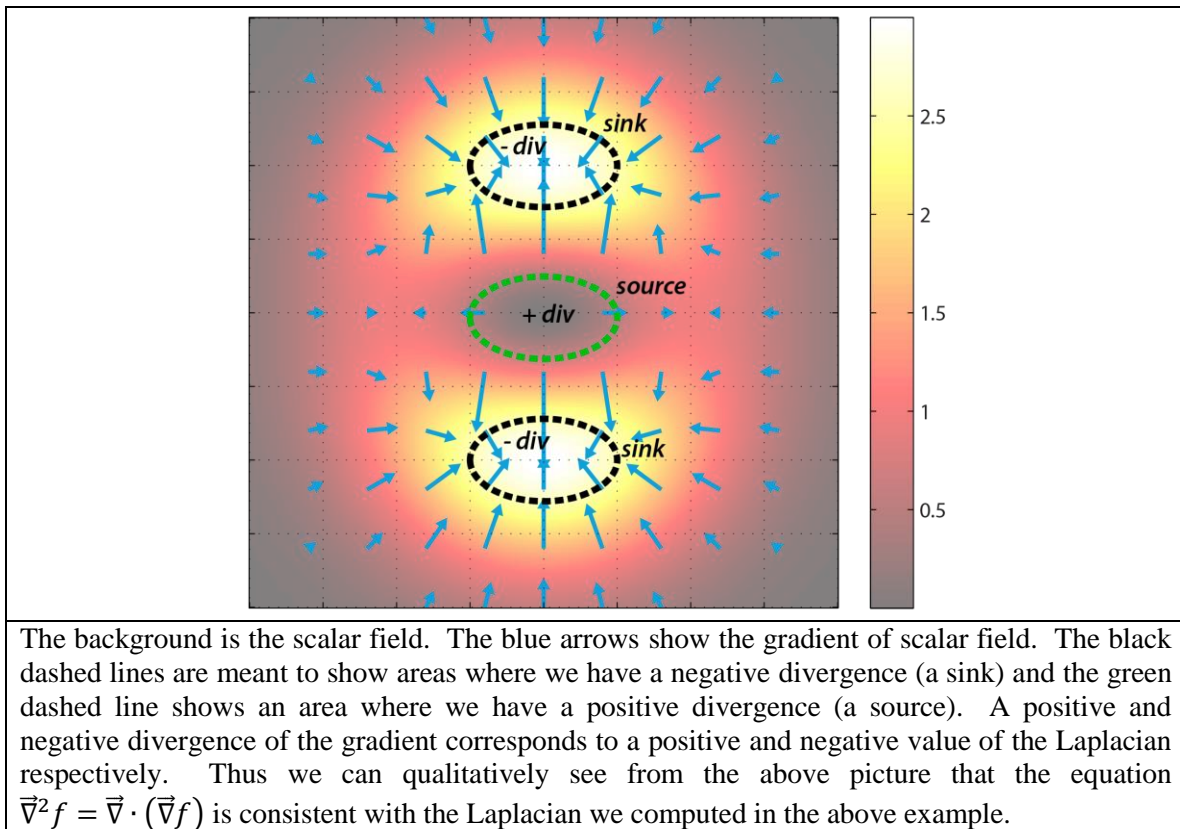


Thus, we see that we have a positive Laplacian (concave up) in the center hole in elevation, and we have a negative Laplacian (concave down) in the peaks in elevation.

Another way to look at the Laplacian is to recall that it is the divergence of the gradient of the function:

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

We calculated and plotted the gradient of the function discussed above already. So, we can qualitatively just look at whether we have sources (positive divergence) or sinks (negative divergence) in the gradient field.



Now, let's consider what this means in an example. For example, the heat conduction equation is written as follows:

$$\nabla^2 T = \alpha \frac{\partial T}{\partial t}$$

Where,

T = temperature

$\alpha$  = thermal diffusivity.



This equation explains how temperature will change with time. So, let's consider our example above. Instead of elevation, let's let this represent an initial temperature field. Thus, where we have peaks, these areas are hotter than the others. Where we have depressions, these areas are cooler than the others. What does the above equation tell us:

- Peaks (**hotter regions**) - here the Laplacian is negative. Thus,

$$\Delta T = \frac{1}{\alpha} (\vec{\nabla}^2 T) \Delta t$$

$$\Rightarrow \Delta T = \frac{1}{\alpha} (\text{negative number}) \Delta t$$

Thus, in these regions the change in temperature will be negative. That is, the peak will start to cool down. This makes sense, as the surrounding points are cooler.

- Troughs (**cooler regions**) – here the Laplacian is positive.

$$\Rightarrow \Delta T = \frac{1}{\alpha} (\text{positive number}) \Delta t$$

Thus, in these regions the change in temperature will be positive. That is, the trough will start to heat up as the surrounding points are hotter.

Many geophysical potential fields, such as gravitational potential, satisfy Laplace's equation when there are no sources included, and are referred to as Laplacian fields:

Laplace's Equation:

$$\vec{\nabla}^2 \phi = 0$$

A function that satisfies Laplace's Equation is referred to as a **harmonic function**.

Laplace's equation is exceptionally useful in geophysics and we will encounter it often! However, it should be noted that many non-Laplacian fields are also commonly used in geophysics. These fields are referred to as **self-potential fields**.

#### 4. Linear Transformations

A linear operator acts on a vector to produce a new vector:

$$A\vec{u} = \vec{v}$$

Where,  $A$  is the operator.

For the operator to be linear the following property must be satisfied:

$$A(\alpha\vec{u} + \beta\vec{v}) = \alpha A\vec{u} + \beta A\vec{v}$$

It's easiest to understand these operators by looking at some examples.

#### 4.1 Example - Rotation Matrix

Let's consider an operator,  $R$ , that rotates a vector in 2-D. The operator is defined as:

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Let this operate on the vector:  $\vec{u} = 1\hat{i} + 0\hat{j}$ .

Thus,

$$\vec{v} = A\vec{u}$$

Writing our vectors as column vectors:

$$\Rightarrow \begin{bmatrix} \vec{v}_x \\ \vec{v}_y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{v}_x \\ \vec{v}_y \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

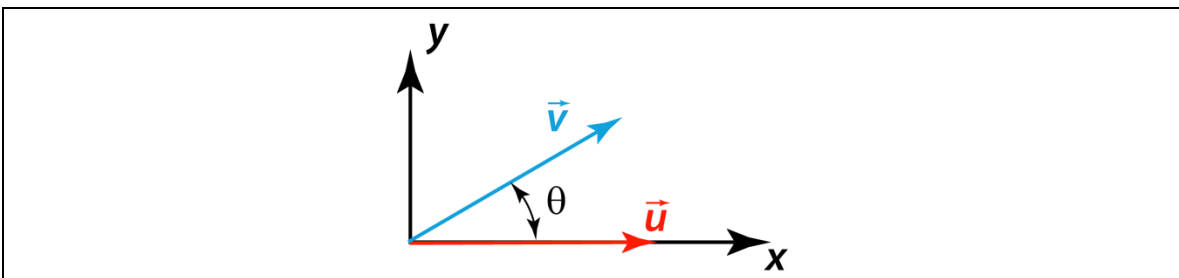
$$\Rightarrow \vec{v} = \cos \theta \hat{i} + \sin \theta \hat{j}$$

Let's take a practical example where we let  $\theta=30^\circ$ . Then,

$$\Rightarrow \vec{v} = \cos 30^\circ \hat{i} + \sin 30^\circ \hat{j}$$

$$\Rightarrow \vec{v} = \frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j}$$

Graphically,  $\vec{u}$  gets rotated counter-clockwise by  $30^\circ$  into the new vector  $\vec{v}$ .



This is useful in many cases, and can be used to rotate coordinate systems. For example, in seismology we may want to rotate our displacement vectors out of a geographic coordinate system (N-S, E-W) into a coordinate system along the great circle path.

#### 4.2 Some other useful operators

Some other useful transformations (in 2D) are as follows:

<b>Operator</b>	<b>Effect</b>
$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection about the $x$ -axis
$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection about the $y$ -axis
$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	Scaling by 2 in all directions. Obviously, can be generalized to different scaling.
$A = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$	Horizontal shear mapping
$A = \begin{bmatrix} k & 0 \\ 0 & 1/k \end{bmatrix}$	Squeeze mapping