Development of Rotation in Supercell Thunderstorms

See Holton pgs 299-304. I will use tensor notation to simplify the algebra (derivation for grad students is important)

We'll use the anelastic continuity eqn $\bar\rho \frac{\partial u}{\partial x} + \bar\rho\frac{\partial v}{\partial y} + \frac{\partial(\bar\rho w)}{\partial z} = 0$ C5.1

or: $\frac{1}{\bar\rho}\frac{\partial \bar\rho w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$ C5.2

Repeating C1.2 $\frac{d\vec V}{dt} = \frac{\partial \vec V}{\partial t} + \vec V\cdot\nabla\vec V = -\frac{1}{\bar\rho}\nabla p' + b\hat k$ C5.3

In tensor notation $\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\bar\rho}\frac{\partial p'}{\partial x_i} + \delta_{i3}\, b$ C5.4

where $\delta_{i3} = 1$ if $i=3$ & $=0$ if $i=1$ or 2

We're interested in how rotation (especially about the vertical axis) develops in a thunderstorm. So we need to understand $\frac{d\vec\omega}{dt}$

$\nabla\times$ C5.3 or $\epsilon_{kli}\frac{\partial}{\partial x_l}$ C5.4 is what we're after. Let's do the vertical component of vorticity first. $\omega_3 = \epsilon_{3li}\frac{\partial u_i}{\partial x_l}$

So, $\frac{\partial}{\partial t}\left[\epsilon_{3li}\frac{\partial u_i}{\partial x_l}\right] + u_j\frac{\partial}{\partial x_j}\left[\epsilon_{3li}\frac{\partial u_i}{\partial x_l}\right] + \epsilon_{3li}\frac{\partial u_j}{\partial x_l}\frac{\partial u_i}{\partial x_j} = -\frac{\epsilon_{3li}}{\bar\rho}\frac{\partial}{\partial x_l}\frac{\partial p'}{\partial x_i} + \epsilon_{3li}\delta_{i3}\frac{\partial b}{\partial x_l}$

$$\frac{d\omega_3}{dt} = -\epsilon_{3li}\frac{\partial u_j}{\partial x_l}\frac{\partial u_i}{\partial x_j} \underbrace{- \frac{\overset{+1}{\cancel{\epsilon_{312}}}}{\bar\rho}\frac{\partial}{\partial x_1}\frac{\partial p'}{\partial x_2} - \frac{\overset{-1}{\cancel{\epsilon_{321}}}}{\bar\rho}\frac{\partial}{\partial x_2}\frac{\partial p'}{\partial x_1}}_{0} + \underset{0}{\cancel{\epsilon_{3l3}}}\,\overset{1}{\cancel{\delta_{33}}}\frac{\partial b}{\partial x_l}$$

$$= -\overset{-1}{\cancel{\epsilon_{321}}}\frac{\partial u_j}{\partial x_2}\frac{\partial u_1}{\partial x_j} - \underset{1}{\cancel{\epsilon_{312}}}\frac{\partial u_j}{\partial x_1}\frac{\partial u_2}{\partial x_j}$$

$$= \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}$$

$$= -\underbrace{\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]}_{\omega_3}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}$$ C5.5

Plugging in from continuity C5.2 :

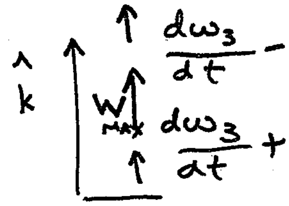
$$\frac{d\omega_3}{dt} = \frac{\omega_3}{\bar{\rho}}\frac{\partial \bar{\rho} W}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} \qquad \text{C6.1}$$

stretching term tilting terms

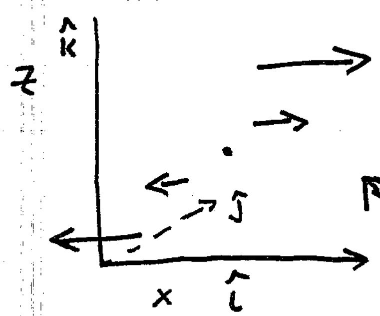
Initially, stretching term can't generate vertical vorticity unless $\omega_3 \neq 0$

So, if we have cc vertical vorticity (ω_3 +), then rotation increases in regions where $\frac{\partial W}{\partial z}$ + (which is below maximum updraft)

So, the stretching term can increase cc rotation <u>below</u> max updraft, once rotation present

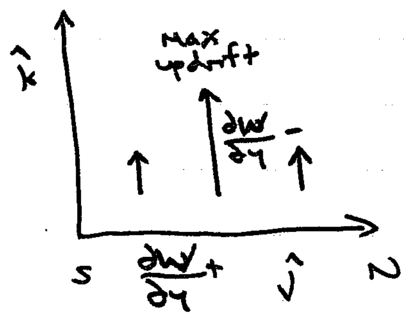


Now, consider the tilting terms. Consider simple case where $v=0$, & $\frac{\partial u}{\partial z}$ +



$$\omega_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \quad (+)$$

(direction of rotation into board/page)



In the $\hat{j}, \hat{k}$ plane, assume the following

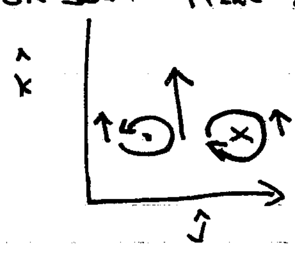
On south flank of updraft $\frac{\partial w}{\partial y}$ +

" north " " " $\frac{\partial w}{\partial y}$ −

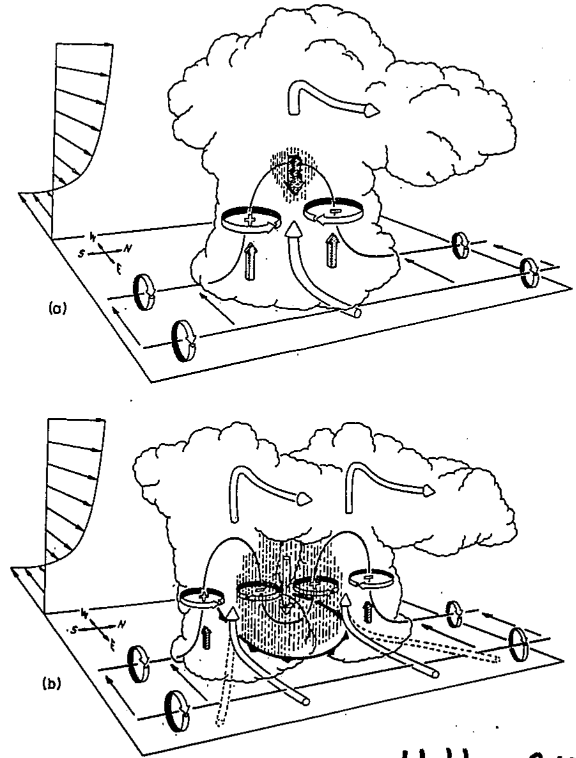
$\frac{d\omega_3}{dt} \propto \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}$ So $\frac{d\omega_3}{dt}$ increases on south flank of updraft

decreases " north " " "

∴ storm is acquiring cc rotation on south flank & clockwise rotation on north flank



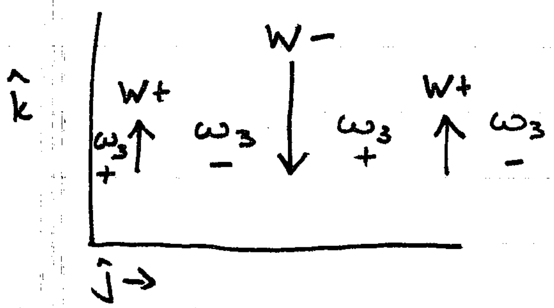
As shown in (a), initially the convection updraft causes cc rotation forming on south flank as a result of the tilting of the environmental shear. Visualize the thin solid lines as "material" vortex tubes lifted & deformed by the convective updraft



Holton 9.12

Development of rotation and splitting in a supercell storm with westerly mean wind shear (shown by storm relative wind arrows in the upper left corner of each panel). Cylindrical arrows show the direction of cloud relative air flow. Heavy solid lines show vortex lines with a sense of rotation shown by circular arrows. Plus and minus signs indicate cyclonic and anticyclonic rotation caused by vortex tube tilting. Shaded arrows represent updraft and downdraft growth. Vertical dashed lines denote regions of precipitation. (a) In the initial stage, the environmental shear vorticity is tilted and stretched into the vertical as it is swept into the updraft. (b) In the splitting stage, downdraft forms between the new updraft cells. Barbed line at surface indicates downdraft outflow at surface. (After Klemp, 1987.)

Once a downdraft forms in the core (b), splitting of the storm develops. One cell may preferentially develop relative to the other as a result of the distribution of pressure.



I was a little loose in the original derivation on pg. 11. The full diagnostic relationship for the pressure perturbations is:

$$\nabla^2 p' = \frac{\partial \bar{\rho} b}{\partial z} - \nabla \cdot (\bar{\rho}\vec{V} \cdot \nabla \vec{V}) \qquad \text{C7.1}$$

3 dimensional Laplacian of pressure perturbation $\sim -p'$

vertical gradient of buoyancy

advective effects important in tornadoes & outflows

Focusing on the horizontal distribution of perturbation pressure

$$\nabla_h^2 p' = -\nabla_h \cdot (\bar{\rho}\vec{V}_h \cdot \nabla_h \vec{V}_h) \qquad \underline{C8.1}$$

$$-\frac{p'}{\bar{\rho}} \sim -\frac{\partial}{\partial x_i} u_j \frac{\partial u_i}{\partial x_j} \Rightarrow \frac{p'}{\bar{\rho}} = \frac{\partial u_j}{\partial x}\frac{\partial u}{\partial x_j} + \frac{\partial u_j}{\partial y}\frac{\partial v}{\partial x_j}$$

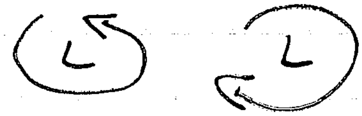
$$\frac{p'}{\bar{\rho}} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} = \cancel{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Ignore from scaling arguments

If the rotation is symmetric then $\omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

or $\frac{\partial v}{\partial x} = \frac{1}{2}\omega_3$ & $\frac{\partial u}{\partial y} = -\frac{1}{2}\omega_3$

Then $\frac{p'}{\bar{\rho}} \sim \frac{1}{2}\omega_3^2$ $\qquad \underline{C8.2}$



This is important: when strong rotation present, there is always low pressure at the center (experiment: stir coffee cup)
For convective motions, low pressure is associated with both clockwise & cclockwise flow

Finally, we want to be able to explain why vortex on right flank of updraft tends to develop preferentially.

Let $\vec{S} = \frac{\partial \vec{V}_h}{\partial z} = \frac{\partial u}{\partial z}\hat{i} + \frac{\partial v}{\partial z}\hat{j}$ = wind shear vector

Then C6.1 can be written $\frac{d\omega_3}{dt} = \frac{\omega_3}{\bar{\rho}}\frac{\partial \bar{\rho} w}{\partial z} + \hat{k}\cdot\vec{S}\times\nabla_h w$ $\qquad \underline{C8.3}$

& $\vec{\omega}_h = \hat{k}\times\vec{S}$ if the vertical shear of the wind is small (see pg C3) $\qquad \underline{C8.4}$

Then C7.1 can be written $\nabla^2 p' = \frac{\partial \bar{\rho} b}{\partial z} - \frac{\partial}{\partial z}(\bar{\rho}\vec{V}_h \cdot \nabla_h w)$ $\qquad \underline{C8.5}$

i.e. that the vertical advection term dominates over the horizontal ones

One more handwaving: assume $\frac{\partial}{\partial z}[\bar{\rho}\vec{V}_h \cdot \nabla_h w] \sim \bar{\rho}\frac{\partial \vec{V}_h}{\partial z} \cdot \nabla_h w$
(that is that the vertical shear of the vertical wind is small)

$$\text{Then } \frac{1}{\bar{\rho}}\nabla^2 p' = \frac{\partial b}{\partial z} - \vec{S} \cdot \nabla_h w \qquad \text{C9.1}$$

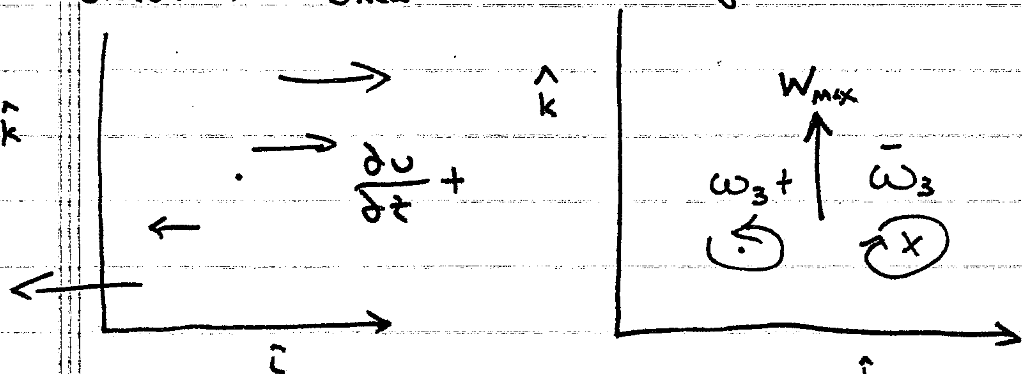
$\frac{\partial b}{\partial z}$: results in pressure perturbations due to buoyancy

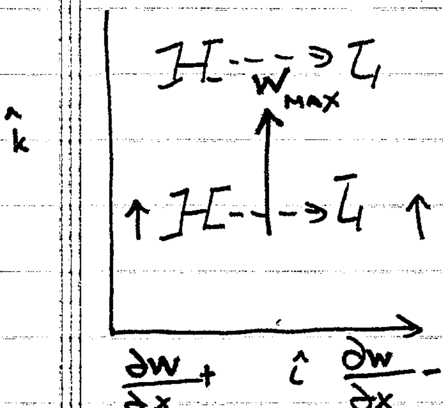
$\vec{S} \cdot \nabla_h w$: results in pressure perturbations due to "dynamics"

$$\text{or } \frac{p'_{dyn}}{\bar{\rho}} \sim \vec{S} \cdot \nabla_h w \qquad \text{C9.2}$$

Assume $\vec{S} = \frac{\partial u}{\partial z}\hat{i}$ Then $\frac{p'_{dyn}}{\bar{\rho}} \sim \frac{\partial u}{\partial z}\frac{\partial w}{\partial x}$
unidirectional shear

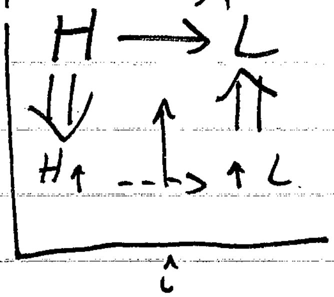
Repeating from earlier





This will lead to favorable development for development of next cell on east flank of updraft where perturbation low pressure is located

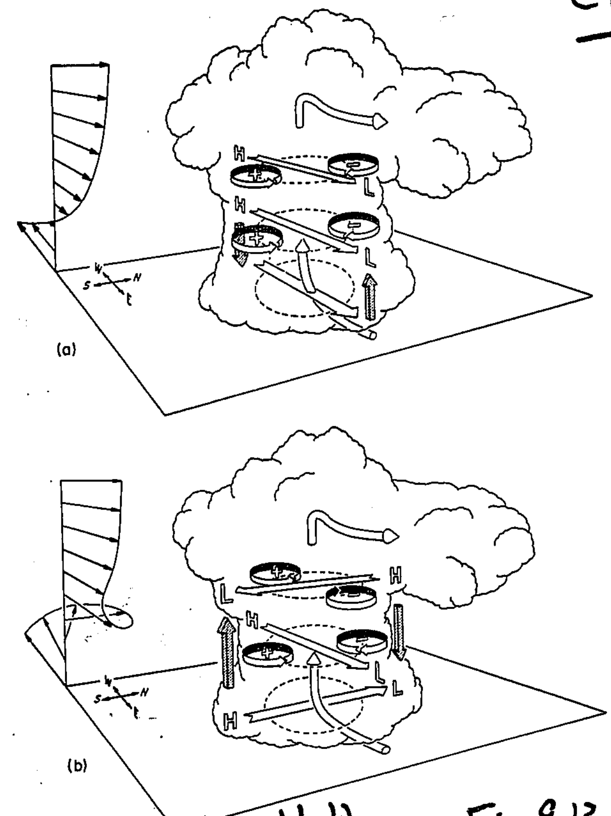
Now since shear & vertical velocity contrasts likely to be stronger aloft, pressure perturbations aloft typically stronger as well



Then that drives vertical pressure perturbations as shown, which contribute to rising motion on east flank

C10

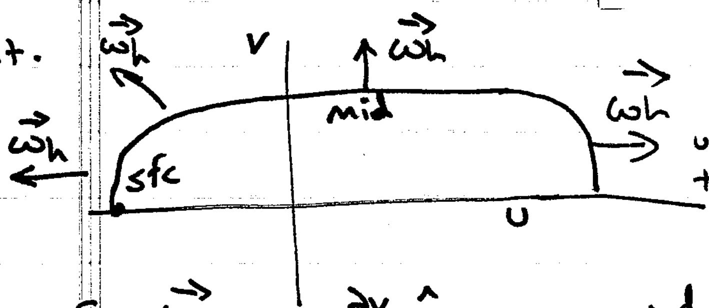
(a) summarizes this situation of unidirectional wind shear & shows that the developing updrafts (east flank) & cc rotation (south flank) are not lined up. As these storms develop in unidirectional shear, it is likely that splitting will occur, without preferential development of one cell or another.



Holton Fig 9.13

Pressure and vertical vorticity perturbations produced by interaction of the updraft with environmental wind shear in a supercell storm. (a) Wind shear does not change direction with height. (b) Wind shear turns clockwise with height. Broad open arrows designate the shear vectors. H and L designate high and low dynamical pressure perturbations, respectively. Shaded arrows show resulting disturbance vertical pressure gradients. (After Klemp, 1987.)

(b) summarizes the common situation of clockwise turning of the horizontal wind with height.



Near ground $\vec{S} = +\frac{\partial v}{\partial z}\hat{j}$

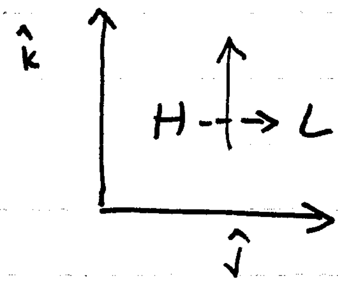
mid trop $\vec{S} = \frac{\partial u}{\partial z}\hat{i}$

upper trop $\vec{S} = -\frac{\partial v}{\partial z}\hat{j}$

So, $\vec{\omega_h} = -\frac{\partial v}{\partial z}\hat{i}$ near ground, etc.

From C9.1 $\frac{1}{\bar{\rho}}\nabla^2 p'_{dyn} = -\vec{S}\cdot\nabla_h w \sim \frac{p'_{dyn}}{\bar{\rho}}$

So, near ground $\frac{p'_{dyn}}{\bar{\rho}} \sim \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}$

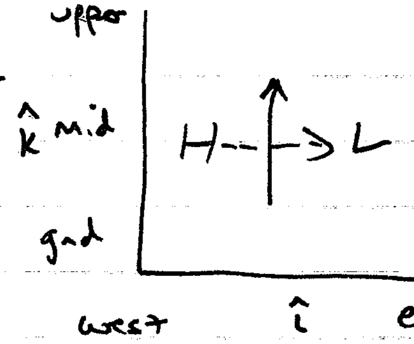


near ground

Drives flow from south flank to north flank near ground across updraft

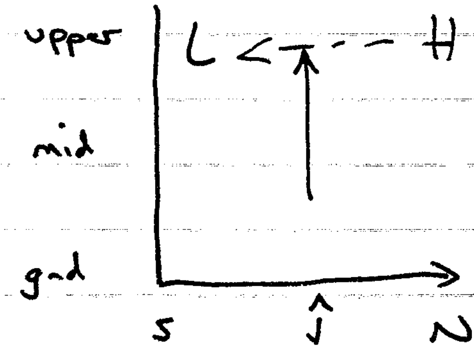
Mid troposphere: $\dfrac{p'_{dyn}}{\bar{\rho}} \sim \dfrac{\partial u}{\partial z}\dfrac{\partial w}{\partial x}$

Drives flow across updraft in mid levels from west to east



Upper troposphere $\dfrac{p'_{dyn}}{\bar{\rho}} \sim -\dfrac{\partial v}{\partial z}\dfrac{\partial w}{\partial y}$

Drives flow across updraft in upper troposphere from north to south



See Holton Fig 9.13(b) (previous page). Net result is that the combination of low level positive pressure perturbation & upper level negative pressure perturbation can foster preferentially updrafts on south flank of existing system. This is to right of motion of the cells which is moving in direction of mean wind through sfc-mid levels. And, this is where mesocyclone is preferentially forming as well!

Complicated: yes! But, an instructive use of basic dynamics to understand what causes storms to form, split, & preferential development. This approach does not explain continued contraction & spin up of tornadoes.