LES of Turbulent Flows: Lecture 12
(ME EN 7960-008)

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Dynamic SGS models

• So far we have given a general description of some commonly used SGS models

• **All of these models include at least one model coefficient** that must be prescribed either based on theory with a specific set of assumptions (usually isotropy), from experimental data, or chosen *ad hoc* to get the "correct" *a posteriori* results from simulations.

• Germano et al. (PofF, 1991) developed a procedure to dynamically calculate these unknown model coefficients (for scalars and compressible flow see Moin et al., PofF, 1991).

• Recall: Applying a low-pass filter to the N-S equations with a filter of characteristic width Δ (denoted by ~) results in the unknown SGS stress term:

$$\tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j \quad \text{(a)}$$

• This term must be modeled with an SGS model to close our equation set.

Dynamic SGS models

- We can apply another filter (referred to as a test filter) to the filtered N-S equations at a larger scale (say 2Δ) denoted by a bar (**–**):

$$\overline{\frac{\partial \tilde{u}_i}{\partial t}} + \overline{\frac{\partial \tilde{u}_i \tilde{u}_j}{\partial x_j}} = -\overline{\frac{\partial \tilde{p}}{\partial x_i}} + \overline{\frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2}} - \overline{\frac{\partial \tau_{ij}}{\partial x_j}} + F_i$$

- Our LES filter properties (commutation with differentiation) allows us to rewrite the 1st term on the LHS and the 1st and 2nd terms on the RHS in standard filtered form.
- The convective term can be reformatted into our standard format using the same method we used for the original filtered LES equations (see Lecture 5):

$$\overline{\frac{\partial \tilde{u}_i \tilde{u}_j}{\partial x_j}} = \frac{\partial}{\partial x_j}\left(\overline{\tilde{u}_i \tilde{u}_j} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j + \bar{\tilde{u}}_i \bar{\tilde{u}}_j\right) = \frac{\partial \bar{\tilde{u}}_i \bar{\tilde{u}}_j}{\partial x_j} + \frac{\partial \overline{\tilde{u}_i \tilde{u}_j} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j}{\partial x_j}$$

- where the 1st term is our standard form and we can move the 2nd term to the RHS of our twice filtered equation and combine it with the SGS force vector $\overline{\frac{\partial \tau_{ij}}{\partial x_j}}$ to form our test filtered SGS stress:

$$\underbrace{-\overline{\widetilde{u_i u_j}} + \overline{\tilde{u}_i \tilde{u}_j}}_{\bar{\tau}_{ij}} \underbrace{- \overline{\tilde{u}_i \tilde{u}_j} + \bar{\tilde{u}}_i \bar{\tilde{u}}_j}_{\text{from the convective term}}$$

- Our SGS stress at the 2Δ level can now be written:

$$\boxed{T_{ij} = \overline{\widetilde{u_i u_j}} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j}$$ (b)

Dynamic SGS models

• We can also consider the stress at our smallest resolved scales (the Leonard stress we discussed in Lecture 11):

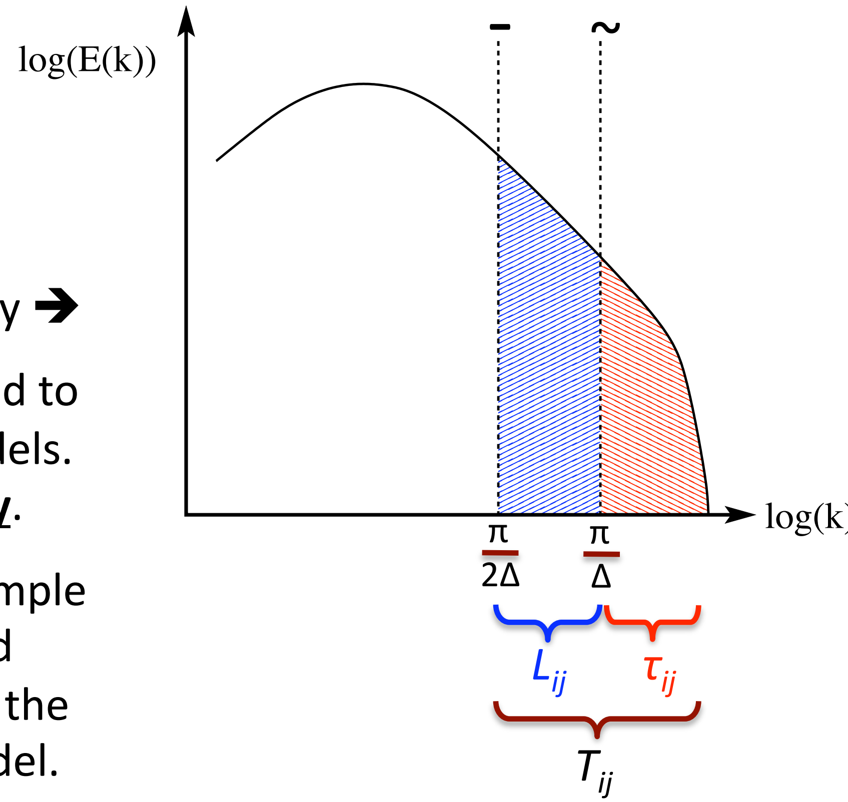
$$Lij = \overline{\tilde{u}_i\tilde{u}_j} - \bar{\tilde{u}}_i\bar{\tilde{u}}_j \quad \text{(c)}$$

• Equations a, b and c can be combined algebraically as follows:

$$L_{ij} = T_{ij} - \bar{\tau}_{ij} \quad (*)$$

$$\overline{\tilde{u}_i\tilde{u}_j} - \bar{\tilde{u}}_i\bar{\tilde{u}}_j = \overline{\widetilde{u_i u_j}} - \bar{\tilde{u}}_i\bar{\tilde{u}}_j - \overline{\widetilde{u_i u_j}} + \overline{\tilde{u}_i\tilde{u}_j}$$

Graphically, this equation can be represented by ➔



• (*) is an identity, **it is exact**! It can be exploited to derive model coefficients for common SGS models. It is usually referred to as the **Germano identity**.

• We will use the Smagorinsky model as an example of how we can use the Germano identity to find model coefficients. Procedurally, we can follow the same steps presented next for any bas SGS model.

Dynamic Smagorinsky Model

- The first assumption we must make is that the same model (e.g. Smagorinsky model) can be applied for the stress at Δ and αΔ (or 2Δ).
- Using the Smagorinsky model in the Germano identity (Note Smagorinsky is only for anisotropic part):

$$L_{ij} - \frac{1}{3}\delta_{ij}L_{kk} = T_{ij} - \bar{\tau}_{ij}$$

$$\overline{\tilde{u}_i\tilde{u}_j} - \bar{\tilde{u}}_i\bar{\tilde{u}}_j = -2\left(C_S\alpha\Delta\right)^2|\bar{\tilde{S}}|\bar{\tilde{S}}_{ij} + \overline{2\left(C_S\alpha\Delta\right)^2|\tilde{S}|\tilde{S}_{ij}}$$

- For the next parts we will follow Lilly (PofF, 1992)
- We can rearrange this equation to write an equation for the error associated with using the Smagorinsky model in the Germano identity.

$$e_{ij} = L_{ij} - \frac{1}{3}\delta_{ij}L_{kk} - \left[-2\left(C_S\alpha\Delta\right)^2|\bar{\tilde{S}}|\bar{\tilde{S}}_{ij} + 2\overline{\left(C_S\alpha\Delta\right)^2|\tilde{S}|\tilde{S}_{ij}}\right]$$

- This can be rewritten as (note we will assume L_{ij} is trace free)

$$e_{ij} = L_{ij} - C_S^2 M_{ij}$$

$$\rightarrow M_{ij} = 2\Delta^2\left[\overline{|\tilde{S}|\tilde{S}_{ij}} - \alpha^2|\bar{\tilde{S}}|\bar{\tilde{S}}_{ij}\right]$$

- Problem: This is 9 equations with only 1 unknown!

Dynamic Smagorinsky Model

- Lilly (1992) proposed to minimize the error in a least-squares sense. That is, we want the least-square error of using the Smagorinsky model in the Germano identity.
- The squared error is: $e_{ij}^2 = \left(L_{ij} - C_S^2 M_{ij}\right)^2 = L_{ij}^2 - 2C_S^2 L_{ij} M_{ij} + (C_S^2)^2 M_{ij} M_{ij}$

We want the minimum with respect to C_S^2, i.e.: $\dfrac{\partial e_{ij}^2}{\partial C_S^2} = 0$

=> $$\frac{\partial e_{ij}^2}{\partial C_S^2} = -2L_{ij} M_{ij} + 2C_S^2 M_{ij} M_{ij} = 0$$

If we solve for C_S^2 => $$C_S^2 = \frac{L_{ij} M_{ij}}{M_{ij} M_{ij}}$$

- Problem: the above local form of the dynamic Smagorinsky coefficient is numerically unstable. Remember the energy cascade can be forward or backwards instantaneously. In simulations this was found to lead to numerical instability (having $\pm C_S^2$).
- The instability is attributed to high time correlations of C_S^2 (i.e., when C_S^2 is negative at a point it tends to stay negative)
- Why do we have this problem? **We had to make 2 assumptions to derive the C_S^2 equation**!

Assumptions in the Dynamic Model

- **1st Assumption**:

  - C_S^2 is constant over the filter width αΔ (**–** filter in the equations)

  Recall our basic definition of a convolution filter: $\widetilde{\phi}(\vec{x},t) = \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta}$

  If look at our error equation (shown below), we notice that C_S^2 falls under our bar filter:

$$e_{ij} = L_{ij} - \frac{1}{3}\delta_{ij}L_{kk} - \left[-2\left(C_S\alpha\Delta\right)^2 |\bar{\tilde{S}}|\bar{\tilde{S}}_{ij} + \overline{2\left(C_S\alpha\Delta\right)^2 |\tilde{S}|\tilde{S}_{ij}}\right]$$

  ⇒This is actually **a set of integral equations** if we don't make our assumption!

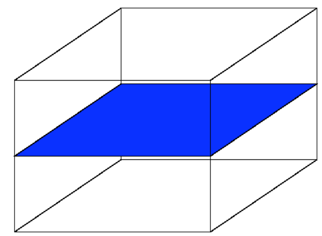
- Ghosal et al. (JFM, 1995) solved this equation for C_S^2 everywhere using a variational method. This is very expensive and complex.
- The constant C_S^2 with respect to the test filter assumption contributes to the instability discussed previously.
- The typical method (instead of Ghosal et al's method) is to enforce the Germano identity in an average sense:

$$C_S^2 = \frac{\langle L_{ij}M_{ij}\rangle}{\langle M_{ij}M_{ij}\rangle}$$

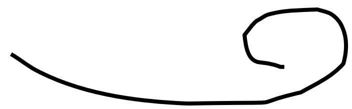
Assumptions in the Dynamic Model

- Constraining C_S^2 removes its oscillations resulting in stable simulations. Typically, **the average is enforced over some region of spatial homogeneity**. For example in a homogeneous boundary layer over horizontal planes:

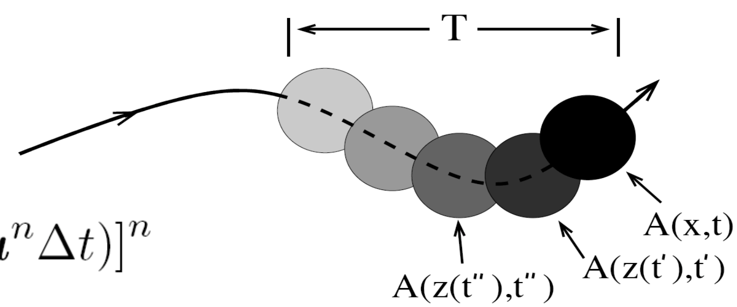
$\langle\,\rangle$ is an averaging operator e.g.
C_S^2 varies only with wall normal direction



- Meneveau et al. (JFM, 1996) developed the Lagrangian Dynamic model based on the idea that the Germano identity should be enforced along fluid particle trajectories =>



Illistration of Lagrangian averaging



-Using 1st order time and space estimates, the average of any quantity A (e.g., L_{ij}) can be defined as:

$$\langle A(\boldsymbol{x})\rangle^{n+1} = \gamma\,[A(\boldsymbol{x})]^{n+1} + (1-\gamma)\,[A(\boldsymbol{x}-\boldsymbol{u}^n\Delta t)]^n$$

where $\gamma \equiv \dfrac{\Delta t/T^n}{1+\Delta t/T^n}$ and T is the Lagrangian timescale that controls how far back in time the average goes (see the figure to the right)

Assumptions in the Dynamic Model

- **2nd Assumption**:

  - When we applied the Smagorinsky model to the Germano identity at 2 different scales, **we made the assumption that the same C_S^2 applies at both scales:**

  i.e. we assumed $C_S^2(\Delta) = C_S^2(2\Delta)$ or scale invariance of C_S^2

  -This assumption is not bad provided that both of our filter scales Δ and 2Δ are in the inertial subrange of turbulence.

  - For cases with at least 1 direction of flow anisotropy we will violate this assumption in some regions of the flow (e.g., near the wall in a boundary layer when z ≤ 2Δ)
  - Porté-Agel et al., (JFM, 2000) developed a generalized dynamic model where C_S^2 **is a function of scale.**

  - They made the weaker assumption that C_S^2 follows a power law distribution at the smallest resolved scales, e.g.: $\dfrac{C_S^2(2\Delta)}{C_S^2(\Delta)} = \dfrac{C_S^2(4\Delta)}{C_S^2(2\Delta)}$

  So that now in our equation for C_S^2 we have M_{ij} as:

$M_{ij} = 2\Delta^2 \left(\overline{|\tilde{S}|\tilde{S}_{ij}} - \alpha^2\beta|\bar{\tilde{S}}|\bar{\tilde{S}}_{ij} \right)$ with $\dfrac{C_S^2(2\Delta)}{C_S^2(\Delta)} = \beta$ the scale-dependence coefficient.