LES of Turbulent Flows: Lecture 2 (ME EN 7960-008)

Prof. Rob Stoll
Department of Mechanical Engineering
University of Utah

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Kolmogorov's Similarity hypothesis (1941)

• smallest scales receive energy at a rate proportional to the dissipation rate (ϵ)

With this he defined the Kolmogorov scales (dissipation scales):

- length scale: $\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$ $\Rightarrow \frac{\eta}{\ell_o} \sim Re^{-3/4}$
- time scale: $\tau = \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}$ $\Rightarrow \frac{v}{U_o} \sim Re^{-1/4}$
- velocity scale: $v = (\nu\epsilon)^{\frac{1}{4}}$ $\Rightarrow \frac{\tau}{t_o} \sim Re^{-1/2}$

<u>Kolmolgorov also hypothesized:</u>

In Turbulent flow, a range of scales exists at very high Re where statistics of motion in a range ℓ (for $\ell_o >> \ell >> \eta$) have a universal form that is determined only by ϵ (dissipation) and independent of ν (kinematic viscosity).

Spectral representation of turbulence

• Fourier decomposition of a signal => the signal (e.g., velocity) is represented by a series of sine and cosine waves of different amplitudes and wavelengths (in 1D):

$$u(x,t) = \sum_k \hat{u}(k,t) e^{ikx}$$

where *k* is the wavenumber (wavelength $\lambda = 2\pi/k$) (See Pope 6.4 and Appendix D,E,F,G or the handout from Stull 88 for details)

• Fourier transforms are useful to study the energy content of a signal with respect to scale (size of motions). They are also used in numerical methods and many other applications.

• The energy content of a signal can be represented by the Energy spectral density:

$$E(k) \equiv \text{ Energy spectral density } \sim \hat{u}(k,t)\hat{u}(k,t)^*$$

where

$$E(k)dk = \text{t.k.e. contained between } k \text{ and } k + dk$$

and

$$\text{total t.k.e} = \int_0^\infty E(k)dk$$

Spectral representation of turbulence

- What are the implications of Kolmolgorov's hypothesis for E(*k*)?

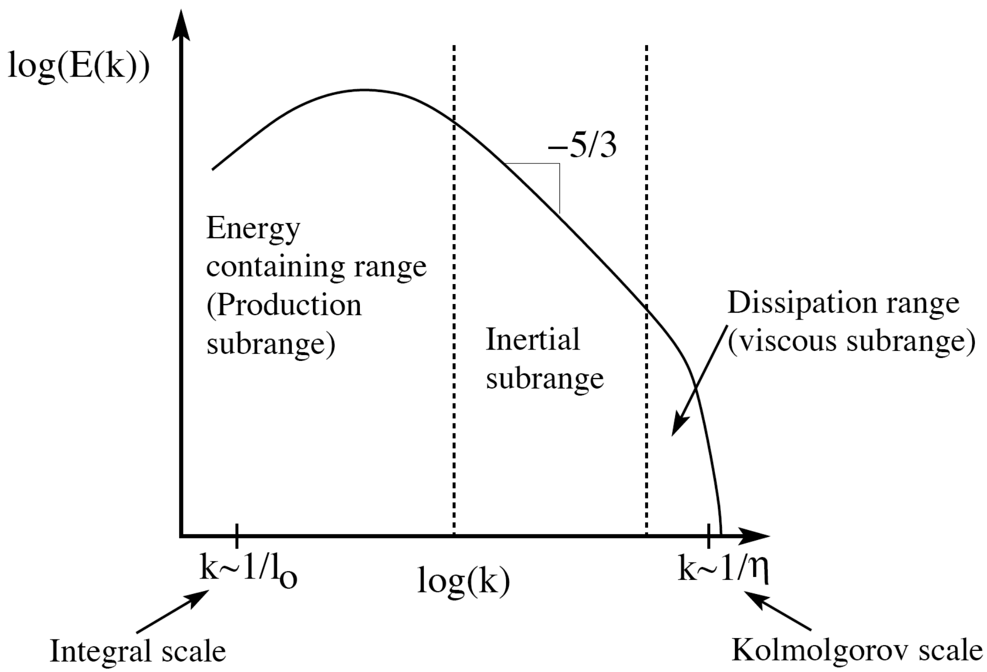
$$\mathrm{K41} \Rightarrow E(k) = f(k, \epsilon)$$

By dimensional analysis we can find that:

$$E(k) = c_k \epsilon^{2/3} k^{-5/3}$$

- This expression is valid for the range of length scales ℓ where $\ell_o >> \ell >> \eta$ and is usually called the inertial subrange of turbulence.

- graphically:



Degrees of freedom and numerical simulations

- We now have a description of turbulence and the range of energy containing scales (the dynamic range) in turbulence

- In CFD we need to discretize the equations of motion (see below) using either difference approximations (finite differences) or as a finite number of basis functions (e.g., Fourier transforms)

- To capture all the dynamics (degrees of freedom) of a turbulent flow we need to have a grid fine enough to capture the smallest **and** largest motions (η and ℓ_o)

- From K41 we know $\frac{\eta}{\ell_o} \sim Re^{-3/4}$ and we have a continuous range of scales between η and ℓ_o

- We need $\frac{\ell_o}{\eta} \sim Re^{3/4}$ in each direction. Turbulence is 3D => we need N~$Re^{9/4}$ points.

Degrees of freedom and numerical simulations

- When will we be able to directly simulate all the scales of motion in a turbulent flow? (Voller and Porté-Agel, 2002, see handouts for the full paper)

In the mid 1960s Gordon Moore, the co-founder of Intel, made the observation that computer power, P, measured by the number of transistors that could be fit onto a chip, doubled once every 1.5 years [1]. This law, which has performed extremely well over the proceeding 30 or so years, can be stated in mathematical terms as

$$P = A2^{0.6667Y}, \quad (1)$$

where A is the computer power at the reference year $Y = 0$.

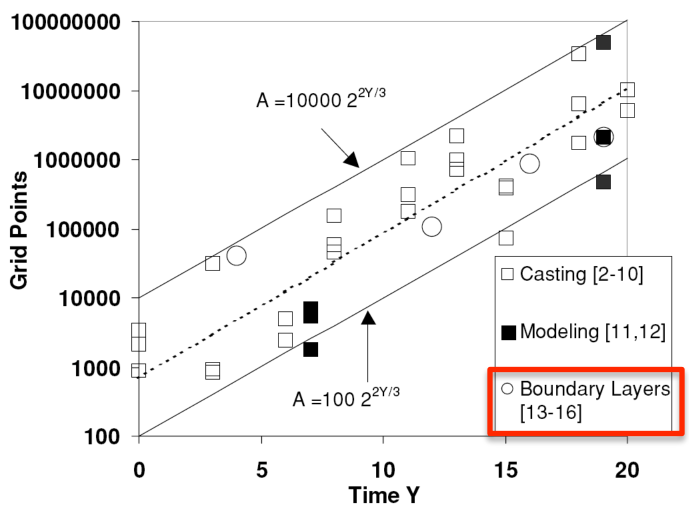


FIG. 1. Log of three largest grid sizes from each volume plotted against year.

TABLE II
Expected Year (±5) That the Given Direct Simulation Will Be Possible If Grid Size Increases Are Bound by Eq. (2)

Simulation	Domain length scale	Resolution length scale	Grid points required	Expected year (±5 years)
2-D casting	0.1 m	1 µm (dendrite tip)	10^{10}	2015
2-D casting	1 m	1 µm (dendrite tip)	10^{12}	2025
3-D casting	0.1 m	1 µm (dendrite tip)	10^{15}	2040
Boundary layer	100 m	1 mm	10^{15}	2040
2-D casting	0.1 m	1 nm (lattice space)	10^{16}	2045
3-D casting	1 m	1 µm (dendrite tip)	10^{18}	2055
2-D casting	1 m	1 nm (lattice space)	10^{18}	2055
Boundary layer	1 km	1 mm	10^{18}	2055
Boundary layer	10 km	1 mm	10^{21}	2070
3-D casting	0.1 m	1 nm (lattice space)	10^{24}	2085
3-D casting	1 m	1 nm (lattice space)	10^{27}	2100

Equations of Motion

• Turbulent flow (and fluid dynamics in general) can be mathematically described by the Navier-Stokes equations (see Bachelor, 1967 for a derivation of equations)

• we use the continuum hypothesis (e.g., $\eta >>$ mean free path of molecules) so that

$$\Rightarrow \quad u_i = u_i(x_j, t) \text{ and } \rho = \rho(x_j, t)$$

• **<u>For incompressible flow:</u>**

-Conservation of Mass (divergent free velocity field):

$$\frac{\partial u_i}{\partial x_i} = 0$$

- Conservation of Momentum:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i$$

Equations of Motion

- If we nondimensionalize these equations with a velocity scale and a length scale (for example the Freestream velocity and the BL height in a boundary layer)

- We get (where the * is a nondimensional quantity):

-Conservation of Mass: $$\frac{\partial u_i^*}{\partial x_i^*} = 0$$

- Conservation of Momentum:

$$\frac{\partial u_i^*}{\partial t^*} + \frac{\partial u_i^* u_j^*}{\partial x_j^*} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i^*}{\partial x_j^{*2}} + F_i^*$$

where Re is based on our velocity and length scales => $Re = \frac{U_o \ell_o}{\nu}$

- For a general scalar quantity we have:

$$\frac{\partial \theta^*}{\partial t^*} + \frac{\partial u_j^* \theta^*}{\partial x_j^*} = \frac{1}{Sc}\frac{\partial^2 \theta^*}{\partial x_j^{*2}} + Q^*$$

where *Sc* is the Schmidt number, the ratio of the diffusivity of momentum (viscosity) and the diffusivity of mass (for temperature we use the Prandtl number *Pr*). Sc is of order 1 (*Pr* for air ≈ 0.72)

Properties of the Navier-Stokes equations

• Reynolds number similarity: For a range of Re, the equations of motion can be considered invariant to transformations of scale.

• Time and space invariance: The equations are invariant to shifts in time or space. i.e., we can define the shifted space variable

$$\hat{x} = \bar{x}/L \text{ where } \bar{x} = x - X$$

$$\text{or } \hat{t} = (t - T)U/L$$

• Rotational and Reflection invariance: The equations are invariant to rotations and reflections about a fixed axis.

• Invariance to time reflections: The equations are invariant to reflections in time. They are the same going backwards or forwards in time =>

$$\hat{t} = -tU/L$$

• Galilean invariance: The equations are invariant to constant velocity translations.

$$\bar{x} = x - Vt$$