LES of Turbulent Flows: Lecture 4
(ME EN 7960-008)

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Scale Separation

- We discussed LES in a very generic way to this point:

  - Resolve only the largest energy containing scales
  - Model the small "universal" scales

- Formally, how is this accomplished?

  - Using a **low-pass filter** (i.e., removes small scale motions)

- Our goal for the low pass filter:

  - Attenuate (smooth) **high frequency** (high wavenumber/small scale) turbulence smaller than a characteristic scale Δ while leaving **low frequency** (low wavenumber/large scale) motions unchanged.

Filtering

- **Filtering** (Saguat chapter 2; Pope chapter 13.2):

  -The formal (mathematical) LES filter is a convolution filter defined for a quantity $\phi(\vec{x},t)$ in physical space as

$$\widetilde{\phi}(\vec{x},t) = \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta}$$

  - $G \equiv$ the convolution kernel of the chosen filter

  - *G* is associated with a characteristic cutoff scale Δ (also called the filter width)

  - Taking the Fourier transform of $\widetilde{\phi}(\vec{x})$ (dropping the t for simplicity)

$$F\{\widetilde{\phi}(\vec{x})\} = \int_{-\infty}^{\infty} e^{-i\vec{k}\vec{x}} \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta})G(\vec{\zeta})d\vec{\zeta}d\vec{x}$$

Here we will use Pope's notation for the Fourier transform: $F\{\phi(x)\} = \int_{-\infty}^{\infty} e^{-ikx}\phi(x)dx$

Convolution

- we can define a new variable: $\vec{r} = \vec{x} - \vec{\zeta}$ and change the order of integration

$$F\{\widetilde{\phi}(\vec{x})\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\vec{k}(\vec{r}+\vec{\zeta})}\phi(\vec{r})G(\vec{\zeta})d\vec{\zeta}d\vec{r}$$

Note that $d\vec{r} = d\vec{x}$ because $\vec{\zeta} \neq f(\vec{x})$ and addition of exponents is multiplication =>

$$F\{\widetilde{\phi}(\vec{x})\} = \int_{-\infty}^{\infty} e^{-i\vec{k}\vec{r}}\phi(\vec{r})d\vec{r} \int_{-\infty}^{\infty} e^{-i\vec{k}\vec{\zeta}}G(\vec{\zeta})d\vec{\zeta}$$

$$\boxed{= F\{\phi(\vec{x})\}F\{G(\vec{\zeta})\}}$$

Sagaut writes this as:

$$\widehat{\widetilde{\phi}}(\vec{k},\omega) = \widehat{\phi}(\vec{k},\omega)\widehat{G}(\vec{k},\omega)$$

where the hat ($\frown$) denotes a Fourier coefficient.

-$\widehat{G}$ is the transfer function associated with the filter kernel G

Recall that a transfer function is the wavespace (Fourier) relationship between the **input** and **output** of a linear system.

Decomposition into resolved and subfilter components

• Just as G is associated with a filter scale Δ (filter width), $\widehat{G}$ is associated with a cutoff wavenumber k_c.

• In a similar manner to Reynold's decomposition, we can use the filter function to decompose the velocity field into resolved and unresolved (or subfilter) components

$$\phi(\vec{x},t) = \widetilde{\phi}(\vec{x},t) + \phi'(\vec{x},t)$$

total resolved subfilter

• **Fundamental properties** of "proper" LES filters:

-The filter **shouldn't change** the value of **a constant** (a):

$$\int_{-\infty}^{\infty} G(\vec{x})d\vec{x} = 1 \;\Rightarrow\; \widetilde{a} = a$$

- **Linearity**: $\widetilde{\phi+\zeta} = \widetilde{\phi} + \widetilde{\zeta}$

(this is satisfied automatically for a convolution filter)

- **Commutation with differentiation**:

$$\widetilde{\frac{\partial \phi}{\partial \vec{x}}} = \frac{\partial \widetilde{\phi}}{\partial \vec{x}}$$

LES and Reynold's Operators

- In the general case, LES filters that verify these properties are not Reynold's operators

    -Recall for a Reynold's operator (average) defined by $\langle\ \rangle$

    - $\langle a\phi\rangle = a\langle\phi\rangle$
    - $\langle\phi'\rangle = 0$
    - $\langle\phi+\zeta\rangle = \langle\phi\rangle + \langle\zeta\rangle$
    - $\langle\langle\phi\rangle\rangle = \langle\phi\rangle$
    - $\langle\frac{\partial\phi}{\partial\vec{x}}\rangle = \frac{\partial\langle\phi\rangle}{\partial\vec{x}}$

- For our LES filter, in general (using Sagauts shorthand $\int_{-\infty}^{\infty}\phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta} = G\star\phi$):
    - $\widetilde{\widetilde{\phi}} = G\star G\star\phi = G^2\star\phi \neq \widetilde{\phi} = G\star\phi$
    - $\widetilde{\phi'} = G\star(\phi - G\star\phi) \neq 0$

- For an LES filter a twice filtered variable is not equal to a single filtered variable as it is for a Reynold's average.

- Likewise, the filtered subfilter scale component is not equal to zero

Differential Filters

- **Differential filters** are a subclass of convolution filter

  - The filter kernel is the Green's function associated to an inverse linear differential operator

  - Recall, the Green's function of a linear differential operator *L* satisfies $L(x)G_r(x,s) = \delta(x-s)$ and can be used to find the solution of inhomogeneous differential equations subject to certain boundary conditions.

- The inverse linear differential operator *J* is defined by:

$$\phi = J(\widetilde{\phi}) = J(G \star \phi)$$

which can be expanded to: $$= \widetilde{\phi} + \theta \frac{\partial \widetilde{\phi}}{\partial t} + \Delta_\ell \frac{\partial \widetilde{\phi}}{\partial x_\ell} + \Delta_{\ell m} \frac{\partial^2 \widetilde{\phi}}{\partial x_\ell \partial x_m}$$

Effectively we need to invert the above equation to define the filter kernel *G*. See Sagaut pgs 20-21 and the references contained therein for more information.

- Differential filters are not used in practice and can be considered an "advanced" topic in LES