LES of Turbulent Flows: Lecture 6
(ME EN 7960-008)

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The filtered kinetic energy equation

• In addition to mass and momentum (see Lecture 5), we also want to look at the **filtered kinetic energy equation** for incompressible flow

- We can find an equation for the filtered kinetic energy by filtering the kinetic energy field:

$$\tilde{E} = \frac{1}{2}\widetilde{u_i u_i}$$

where $\tilde{E}$ is the total filtered kinetic energy.

- The total filtered kinetic energy can be decomposed into a resolved part ($\tilde{E}_f$) and a SFS component ($\tilde{k}_r$):

$$\tilde{E} = \tilde{E}_f + \tilde{k}_r \quad \text{where} \quad \tilde{k}_r = \frac{1}{2}\widetilde{u_i u_i} - \frac{1}{2}\tilde{u}_i\tilde{u}_i$$

Resolved ($\tilde{E}_f$) — SFS ($\tilde{k}_r$)

-An equation for $\tilde{E}_f$ can be derived by multiplying the filtered momentum equation by $\tilde{u}_i$ =>

$$\frac{\partial \tilde{E}_f}{\partial t} + \tilde{u}_j\frac{\partial \tilde{E}_f}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{u}_i\tilde{p}}{\partial x_i} - \frac{\partial \tilde{u}_i\tau_{ij}}{\partial x_j} - 2\nu\frac{\partial \tilde{u}_j\tilde{S}_{ij}}{\partial x_i} - \epsilon_f - \Pi$$

"storage" of $\tilde{E}_f$ | advection of $\tilde{E}_f$ | pressure transport | transport of SFS stress τ_{ij} | transport of viscous stress | dissipation by viscous stress | SFS dissipation

(see Pope pg. 585 or Piomelli et al., Phys Fluids A, 1991)

Transfer of energy between resolved and SFSs

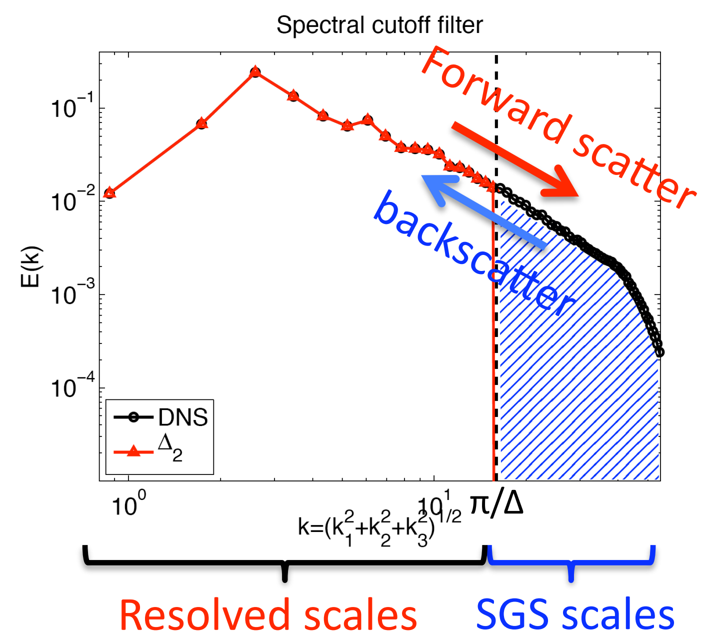
- The **SFS dissipation** Π in the resolved kinetic energy equation is a sink of resolved kinetic energy (it is a source in the $\tilde{k}_r$ equation) and represents the transfer of energy from resolved SFSs. It is equal to:

$$\Pi = -\tau_{ij}\tilde{S}_{ij}$$

- It is referred to as the SFS dissipation as an analogy to viscous dissipation (and in the inertial subrange Π = viscous dissipation).
- On average Π **drains energy** (transfers energy down to smaller scale) from the resolved scales.
- Instantaneously (locally) Π can be positive <u>**or**</u> negative.

  -When Π is negative (transfer from SFS➔Resolved scales) it is typically termed **backscatter**

  -When Π is positive it is sometimes referred to as **forward scatter**.



- Calculating the correct average Π is another necessary (but not sufficient) condition for an LES SFS model (to go with our N-S invariance properties from Lecture 2).

Filtering the compressible N-S equations

- What happens when we apply a filter to the compressible N-S equations?

-**Conservation of Mass for compressible flow:** $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$

- filtering each term => $\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \widetilde{\rho u_i}}{\partial x_i} = 0$ → results in an SFS term!

- How can we avoid having a SFS conservation of mass?

-Density weighted filtering:

- Formalized for compressible flow by Favre (Phys. Fluids, 1983) for ensemble statistics, a Favre (or density weighted) filter is defined by:

$$\bar{\phi} = \frac{\widetilde{\rho\phi}}{\tilde{\rho}} \Rightarrow \tilde{\rho}\bar{\phi} = \widetilde{\rho\phi}$$

where we note that as compressibility becomes less important $\bar{\phi} \rightarrow \tilde{\phi}$
and we can show that the conservation of mass becomes:

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{\rho}\bar{u}_i}{\partial x_i} = 0$$

Filtering the compressible N-S equations

- We can use this to write the Favre filtered equations of motion (see Geurts pg 32-35 or Vreman et al. Applied Sci. Res. 1995 for details)

-**Conservation of Mass:** $\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{\rho}\bar{u}_i}{\partial x_i} = 0$

-**Conservation of Momentum:**

$$\frac{\partial \tilde{\rho}\bar{u}_i}{\partial t} + \frac{\partial \tilde{\rho}\bar{u}_i\bar{u}_j}{\partial x_j} + \frac{\partial \tilde{p}}{\partial x_i} - \frac{\partial \bar{\sigma}_{ij}}{\partial x_j} = -\frac{\partial \tilde{\rho}\tau_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\tilde{\sigma}_{ij} - \bar{\sigma}_{ij}\right)$$

where the SFS terms are collected on the RHS of the equation and we now have both a resolved ($\bar{\sigma}_{ij}$) and SFS ($\tilde{\sigma}_{ij} - \bar{\sigma}_{ij}$) viscous contribution because $\mu = \mu\left(\bar{T}\right)$ is a function of the Favre filtered temperature and

$$\tilde{\sigma}_{ij} = \left(\frac{2}{Re}\right)\widetilde{\mu\left(T\right)\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)}$$ => nonlinear viscous stress tensor

$$\bar{\sigma}_{ij} = \left(\frac{2}{Re}\right)\mu\left(\bar{T}\right)\left(\bar{S}_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial \bar{u}_k}{\partial x_k}\right)$$ => "smooth" viscous stress tensor

Recall: strain rate tensor

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- The SFS stress tensor for the Favre filtered equations is given by

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i\bar{u}_j$$

which is obtained from the nonlinear term => $\widetilde{\rho u_i u_j} = \tilde{\rho}\overline{u_i u_j} = \tilde{\rho}\left(\overline{u_i u_j} - \bar{u}_i\bar{u}_j\right)$

Filtering the compressible N-S equations

-**Conservation of total energy:** $\bar{e} \equiv \text{Favre filtered total energy density} = \frac{\tilde{p}}{\gamma - 1} + \frac{1}{2}\tilde{\rho}\bar{u}_i\bar{u}_i$

$$\frac{\partial \bar{e}}{\partial t} + \frac{\partial}{\partial x_j}\left((\bar{e} + \tilde{p})\,\bar{u}_j\right) - \frac{\partial \bar{u}_i \bar{\sigma}_{ij}}{\partial x_j} + \frac{\partial \bar{q}_j}{\partial x_j} = -a_1 - a_2 - a_3 + a_4 + a_5 - a_6$$

where the LHS contains the SFS terms created using the procedure used in Lecture 5 for τ_{ij}

$$a_1 = \bar{u}_i \frac{\partial \tilde{p}\tau_{ij}}{\partial x_j}$$

⇒ kinetic energy transferred from resolved to SFSs

$$a_2 = \frac{1}{\gamma - 1}\frac{\partial\left(\widetilde{pu_j} - \tilde{p}\bar{u}_j\right)}{\partial x_j}$$

⇒ pressure velocity SFS term (effect of SFS turbulence on the conduction of heat at resolved scales

$$a_3 = \widetilde{p\frac{\partial u_j}{\partial x_j}} - \tilde{p}\frac{\partial \bar{u}_j}{\partial x_j}$$

⇒ compressibility effects (vanishes for incompressible)

$$a_4 = \widetilde{\sigma_{ij}\frac{\partial u_i}{\partial x_j}} - \tilde{\sigma}_{ij}\frac{\partial \bar{u}_i}{\partial x_j}$$

⇒ conversion of SFS kinetic energy to internal energy by viscous dissipation

$$a_5 = \frac{\partial\left(\tilde{\sigma}_{ij}\bar{u}_i - \bar{\sigma}_{ij}\bar{u}_i\right)}{\partial x_j}$$

⇒ SFS viscous stress term

$$a_6 = \frac{\partial\left(\tilde{q}_j - \bar{q}_j\right)}{\partial x_j}$$

⇒ SFS heat flux term (Note q_j is the heat flux vector)

Typically assumptions that $\tilde{\sigma}_{ij} - \bar{\sigma}_{ij} \approx 0$ and $\tilde{q}_j - \bar{q}_j \approx 0$ are made eliminating a_5 and a_6 .