LES of Turbulent Flows: Lecture 9
(ME EN 7960-008)

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Eddy viscosity Models

- **Eddy Viscosity Models:** (Guerts pg 225; Pope pg 587)

-In Lecture 8 we said an eddy-viscosity models are of the form:

momentum: $$\tau_{ij} = -2\nu_T \tilde{S}_{ij}$$

filtered strain rate

eddy-viscosity

scalars: $$q_i = -D_T \frac{\partial \tilde{\theta}}{\partial x_i} \text{ where } D_T = \frac{\nu_T}{Pr_{sgs}}$$

SGS Prandtl number

eddy-diffusivity

-This is the LES equivalent to 1st order RANS closure (k-theory or gradient transport theory) and is an analogy to molecular viscosity (see Pope Ch. 10 for a review)

-Turbulent fluxes are assumed to be proportional to the local velocity or scalar gradients

-In LES this is the assumption that stress is proportional to strain: $\tau_{ij} \sim \tilde{S}_{ij}$

-The SGS eddy-viscosity ν_T must still be parameterized.

Eddy viscosity Models

-Dimensionally =>

$$\nu_T = \left[\frac{L^2}{T}\right]$$

-In almost all SGS eddy-viscosity models $\nu_T \sim u^* l^*$

velocity scale length scale

-Different models use different u^* and l^*

-Recall the filtered N-S equations:

$$\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial \tilde{u}_i \tilde{u}_j}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}}{\partial x_j} + F_i$$

-Note, for an eddy-viscosity model we can write the viscous and SGS stress terms as:

$$\frac{1}{Re}\frac{\partial}{\partial x_j}\tilde{S}_{ij} \quad \text{and} \quad \tau_{ij} = -2\nu_T \tilde{S}_{ij}$$

We can combine these two and come up with a new (approximate) viscous term:

$$\frac{\partial}{\partial x_j}\left[\left(\nu_T + \frac{1}{Re}\right)\tilde{S}_{ij}\right]$$

What does the model do? We can see it effectively lowers the Reynolds number of the flow and for high Re (when 1/Re=>0), it provides all of the energy dissipation.

Smagorinsky Model

•**Smagorinsky model**: (Smagorinsky, MWR 1963)

-One of the 1st and still most popular ν_T models for LES

-Originally developed for general circulation models (large-scale atmospheric), the model did not remove enough energy in this context.

-Applied by Deardorff (JFM, 1970) in the 1st LES.

-Uses Prandtl's mixing length (see Pope Ch. 10 for a review of mixing length) applied at the SGSs:

$$\nu_T = \left(C_S \Delta\right)^2 |\tilde{S}|$$

length scale

velocity scale

-Where Δ is the grid scale taken as $\Delta = \left(\Delta_x \Delta_y \Delta_z\right)^{\frac{1}{3}}$ (Deardorff, 1970 or see Scotti et al., PofF 1993 for a more general description).

-$|\tilde{S}| = \sqrt{2\tilde{S}_{ij}\tilde{S}_{ij}}$ is the magnitude of the filtered strain rate tensor with units [1/T] and serves as the velocity scale (think $\frac{\partial \langle u \rangle}{\partial z}$ in k-theory) and $C_S \Delta$ is our length scale (squared for dimensional consistency).

-The final model is:

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2\left(C_S \Delta\right)^2 |\tilde{S}|\tilde{S}_{ij}$$

-To close the model we need a value of C_S (usually called the Smagorinsky or Smagorinsky-Lilly coefficient)

Lilly's Determination of C_S

- Lilly proposed a method to determine C_S (IBM Symposium, 1967, see Pope page 587)
- Assume we have a high-Re flow => Δ can be taken to be in the inertial subrange of turbulence.
- The mean energy transfer across Δ must be balanced by viscous dissipation, on average (note for Δ in the inertial subrange this is not an assumption) .

$$\epsilon = \langle \Pi \rangle \quad \text{recall:} \quad \Pi = -\tau_{ij}\tilde{S}_{ij}$$

-Using an eddy-viscosity model $\nu_T \Rightarrow \Pi = 2\nu_T \tilde{S}_{ij}\tilde{S}_{ij} = \nu_T |\tilde{S}|^2$

-If we use the Smagorinsky model: $\nu_T = (C_S\Delta)^2 \, |\tilde{S}|$

=> $\Pi = (C_S\Delta)^2 \, |\tilde{S}|^3$

-The square of $|\tilde{S}|$ can be written as (see Pope pg 579 for details):

$$|\tilde{S}|^2 = 2\int_0^\infty k^2 \hat{G}(k)^2 E(k)dk$$

energy spectrum

filter transfer function

-Recall, for a Kolmogorov spectrum in the inertial subrange $E(k) \sim C_k \epsilon^{2/3} k^{-5/3}$

-We can use this in our integral to obtain (see Pope pg. 579): $|\tilde{S}|^2 \approx a_f C_k \epsilon^{2/3} \Delta^{-4/3}$

Lilly's Determination of C_S

- We can rearrange $|\tilde{S}|^2 \approx a_f C_k \epsilon^{2/3} \Delta^{-4/3}$ to get: $\epsilon = \left[\frac{\langle |\tilde{S}|^2 \rangle}{a_f C_k \Delta^{-4/3}} \right]^{\frac{3}{2}}$ (∗)
- Equating viscous dissipation and the average Smagorinsky SGS dissipation ($\epsilon = \langle \Pi \rangle$):

$$\epsilon = \langle (C_S \Delta)^2 \, |\tilde{S}|^3 \rangle$$

- if we now combine this equation with (∗) above and do some algebra…

$$C_S = \frac{1}{(C_k a_f)^{3/4}} \left(\frac{\langle |\tilde{S}|^3 \rangle}{\langle |\tilde{S}|^2 \rangle^{3/2}} \right)^{-\frac{1}{2}}$$

- we can use the approximation $\langle |\tilde{S}|^3 \rangle \approx \langle |\tilde{S}|^2 \rangle^{3/2}$ and a_f for a cutoff filter (see Pope)

=> $$C_S = \frac{1}{\pi} \left(\frac{2}{3C_k} \right)^{3/4}$$

- C_k is the Kolmogorov constant ($C_k \approx$ 1.5-1.6) and with this value we get:

$$\boxed{C_S \approx 0.17}$$